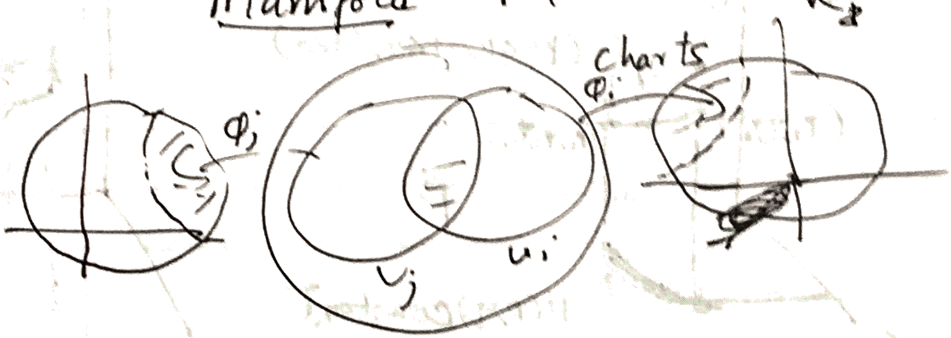
A quick and dirty intro to differential forms.

Manifold M $\qquad$ $\mathbb{R}^n$ $\qquad$ Atlas $\{U_i, \varphi_i\}$ ↑ cover.



The maps $\varphi_i \circ \varphi_j^{-1}$ on appropriate domains $\varphi_j(U_i \cap U_j)$ are C^∞ maps.

Old notation: $\{x^i\}$ $\{x'^i\}$ two different coordinate systems $\dfrac{\partial^n x'^i}{\partial x^{j_1} \cdots \partial x^{j_n}}$

well-defined.

Tangent space at point p.

Old v^i
contravariant vectors: defined through transformations

$$v'^i = v^j \frac{\partial x'^i}{\partial x^j}$$

Modern solution: to think of tangent vectors as directional derivatives of functions

$f \in C^\infty(M, R) = \mathcal{F}(M)$ $f : M \to R$

$v : \mathcal{F}(M) \to R$

$v(f) = v^i \frac{\partial}{\partial x^i} f \Big|_p$ $T_p M$ = all such vectors at p

Note that in different coordinates $v'^i \frac{\partial}{\partial x'^i} = v^j \frac{\partial}{\partial x^j}$

$\Rightarrow$ Transformation rule.

Tangent bundle:

$TM = \{(p, v) \mid p \in M, v \in T_p M\}$

Like the $\{q, \dot{q}\}$ space.

Note that if I have a curve $x^i(t)$

$\frac{dx^i}{dt}$ makes vectors naturally! $\frac{d}{dt} f(x(t)) \Big|_{\text{at } x(t) \to p}$

Vector space V. Dual space V^*: linear maps from V to R.

<u>Differential</u> $df : T_p M \to R$

$v \to v(f)$

Note that df lives in the dual space of $T_p M$, $T_p^* M$.

$\frac{\partial f}{\partial x^i} v^i$

In general: $v \mapsto a_i v^i$ $a \in T_p^*(M)$

Note $dx^i : v \mapsto v^i$

So $a = a_i dx^i$: 1-form

Traditionally covariant vectors defined by

$a'_i = a_j \frac{\partial x^j}{\partial x'^i}$ Consequence of

$a_i \, dx^i = a'_j \, dx'^j$

Cotangent bundle:

$T^*M = \{ \{p, a\} \mid p \in M, a \in T_p^*M \}$

$p_i = \frac{\partial L(q, \dot{q})}{\partial \dot{q}^i}$, $\dot{q} \mapsto p_i \dot{q}$ [redacted]

1 form $p_i \, dq^i$

Tensors: Multilinear maps

$V_1 \otimes \cdots \otimes V_n \to U$

$h: (v_1, \ldots, v_n) \mapsto u$ $\left(\frac{ds}{dt}\right)^2 = g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}$

Example: metric tensor $\rightarrow g_{ij}$ [redacted] $v_2^j = g(v_1, v_2)$

$g: T_pM \otimes T_pM \to \mathbb{R}$

We will focus on covariant anti symetric tensors. They could be thought of as

[redacted]

$\omega : \underbrace{T_pM \otimes \cdots \otimes T_pM}_{n} \to \mathbb{R}$

$\omega(v_1, \dots, v_n)$

[redacted] $= \omega_{i_1 \cdots i_n} v_1^{i_1} \cdots v_n^{i_n}$

Also any index exchange produces -

$$\omega_{\dots i \dots j \dots} = - \omega_{\dots j \dots i \dots}$$

Remember ε_{ijk}. Also we wrote

$$\eta = \begin{pmatrix} q \\ p \end{pmatrix} \qquad \dot{\eta} = J \nabla_\eta H$$

$$\dot{\eta}^i = J^{ij} \frac{\partial H}{\partial \eta^j} \qquad J = \begin{pmatrix} 0 & 1\!\!1 \\ -1\!\!1 & 0 \end{pmatrix}$$

antisymetric [redacted]

Let J_{ij} be the inverse matrix

$$J^{ij} J_{jk} = \delta^i_k$$

J_{jk} is associated with a two form

Wedge product: V and V^*. $a, b \in V^*$, $v_1, v_2 \in V$

$$(a \wedge b)(v_1, v_2) = a(v_1)\, b(v_2) - a(v_2)\, b(v_1)$$

[redacted] Old language $a^i b^j - a^j b^i$: antisymmetrized tensor product

$a: dx^i$

↑ basis of of $T_p^*(M)$

$\Lambda^k(T_p^* M)$

$dx^1 \wedge dx^2 = -dx^2 \wedge dx^1$

$\omega^{(k)} = \frac{1}{k!}\omega_{i_1 \cdots i_k}\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ k-forms

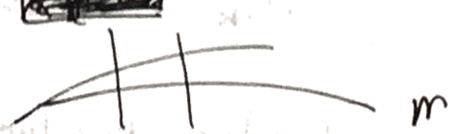
$A \wedge B = A_i dx^i \wedge B_j dx^j = \frac{1}{2}(A_i B_j - A_j B_i) dx^i \wedge dx^j = (A_2 B_3 - A_3 B_2) dx_2 \wedge dx_3 + \cdots$

Vector fields, Tensor fields, Cross product.

[redacted] TM

$\Lambda^k T^* M$ [redacted]

[redacted]



Exterior derivative:

Rules $d(a \wedge b) = da \wedge b + a \wedge db$

$d(da) = 0$

Example: 3 dim

$$df = \frac{\partial f}{\partial x^i}\, dx^i = \bar{\nabla} f \cdot d\vec{r}$$

gradient

$$d(A_i\, dx^i) = d(A_1\, dx^1 + A_2\, dx^2 + A_3\, dx^3)$$

$$= dA_1 \wedge dx_1 + dA_2 \wedge dx^2 + dA_3 \wedge dx^3$$

$$= (\partial_1 A_1\, dx^1 + \partial_2 A_2\, dx^2 + \partial_3 A_3\, dx^3) \wedge dx_1 + \cdots$$

$$dA_i \wedge dx^i = \partial_j A_i\, dx^j \wedge dx^i$$

$$= (\partial_2 A_3 - \partial_3 A_2)\, dx^2 \wedge dx^3 + (\partial_3 A_1 - \partial_1 A_3)\, dx^3 \wedge dx^1$$

$$+ (\partial_1 A_2 - \partial_2 A_1)\, dx^1 \wedge dx^2$$

$$= F_{23}\, dx^2 \wedge dx^3 + F_{31}\, dx^3 \wedge dx^1 + F_{12}\, dx^1 \wedge dx^2$$

$$= (\bar{\nabla} \times \bar{A}) \cdot d\vec{S}$$

$$B^i = \frac{1}{2}\varepsilon^{ijk} F_{jk}$$

$$B^1 = F_{23} = \partial_2 A_3 - \partial_3 A_2$$

$$d(\omega_{ij}\, dx^i \wedge dx^j)$$

$$= \partial_k \omega_{ij}\, dx^k \wedge dx^i \wedge dx^j$$

$$= \partial_k \omega_{ij}\, dx^i \wedge dx^j \wedge dx^k$$

$$(\partial_1 \omega_{23} + \partial_2 \omega_{31} + \partial_3 \omega_{12})\, dx^1 \wedge dx^2 \wedge dx^3$$

If $v^i = \frac{1}{2}\varepsilon^{ijk}\omega_{jk}$

$$(\partial_1 v^1 + \partial_2 v^2 + \partial_3 v^3)\, dx^1 \wedge dx^2 \wedge dx^3$$

$$= \bar{\nabla} \cdot \bar{V}\, d^3\vec{r}$$

divergence

Integration of a k form on k dimensional space

Parametrize C with $x^i(u^1, \dots, u^k)$

$$\int_C \omega = \int \omega_{i_1 \cdots i_k} \frac{\partial x^{i_1}}{\partial u^1} \cdots \frac{\partial x^{i_k}}{\partial u^k}\, du^1 \cdots du^k$$

for small element

$$V \quad \omega\left(\frac{\partial x}{\partial u_1}, \frac{\partial x}{\partial u_2}, \ldots\right) du_1 \cdots du_k$$

Generalized Stokes Theorem

$$\int_{\partial C} \omega = \int_C d\omega$$

Closed forms and exact form

$d\omega = 0$ $\quad\quad$ ω closed

$\omega = d\nu$ $\quad\quad$ ω exact

Since $dd\nu = 0$, all exact forms are closed.
In general, the converse is not true, depends on topology of the manifold.

Example of a ~~phase~~ nonexact 2 form

area form on 2-sphere 

Back to ~~sym~~ dynamics:

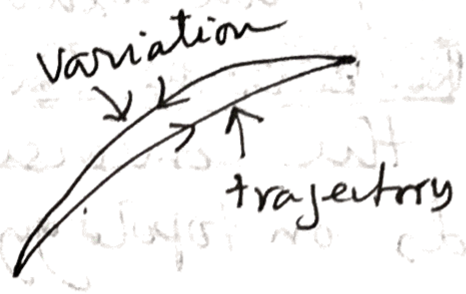
$\lambda = p_i \, dq_i$ 1-form

$\Lambda = p_i \, dq_i - H \, dt$ 1-form

Note $d\lambda = dp_i \wedge dq_i = \omega$

Symplectic form. $= \frac{1}{2} J_{ij} \, d\eta_i \wedge d\eta_j$

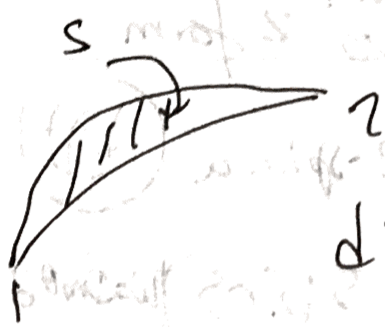
Variational principle



$\oint \Lambda = 0$

Stokes theorem:

$$\int_{\partial S} \Lambda = \int_S d\Lambda$$



$$d\Lambda = dp_i \wedge dq_i - dH \wedge dt = \Omega$$

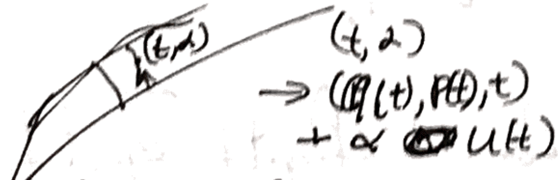
~~Canonical Transformations~~

~~Background~~

$$\Omega = dp_i \wedge dq_i - \frac{\partial H}{\partial q_i} dq_i \wedge dt$$
$$- \frac{\partial H}{\partial p_i} dp_i \wedge dt$$

$$v = (\dot{q}_i, \dot{p}_i, 1)$$

$$\Delta X(t) = u(t)\, \varepsilon$$



$$\int_S \Omega = \int \Omega(u, v)\, d\alpha\, dt$$

If $\Omega(\cdot, v) = 0$, this is zero for any u.

Ω corresponds an antisymmetric $(2n+1)\times(2n+1)$ matrix with entries.

$$|\Omega| = |\Omega^T| = |-\Omega| = (-1)^{2n+1} |\Omega| = -|\Omega|$$

It always has a null vector(?). Let us see if $\vec{v}$ is that null vector, what do we get

$$\begin{pmatrix} 0 & -\mathbb{1} & -\nabla_q H \\ \mathbb{1} & 0 & -\nabla_p H \\ \nabla_q H & \nabla_p H & 0 \end{pmatrix} \begin{pmatrix} \dot{q} \\ \dot{p} \\ 1 \end{pmatrix} = 0$$

$$\dot{p} = -\nabla_q H \qquad \dot{q} = \nabla_p H$$

Last one $\dot{q}_i \partial_{q_i} H + \dot{p}_i \partial_{p_i} H$

$$= \partial_{p_i} H \, \partial_{q_i} H - \partial_{q_i} H \, \partial_{p_i} H = 0$$

Note that $\frac{dH}{dt} = \underbrace{\dot{q}_i \frac{\partial H}{\partial q_i} + \dot{p}_i \frac{\partial H}{\partial p_i}}_{\text{zero}} + \frac{\partial H}{\partial t}$

$$= \frac{\partial H}{\partial t}$$