The Hamilton equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$

$\downarrow$

p_i

$$\{q_i, \dot{q}_i\}$$

$$\rightarrow \{p_i, q_i\}$$

canonical variables

Perhaps:

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \qquad \dot{p}_i = \frac{\partial L}{\partial q_i}$$

Not quite there yet. $dL = \frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial L}{\partial t} dt$

↑ want this guy to be an indept var

Legendre transformation:

$$f(x, y)$$

$$df = u\,dx + v\,dy$$

$$u = \frac{\partial f}{\partial x} \qquad v = \frac{\partial f}{\partial y}$$

$$g = f - ux$$

$$dg = \cancel{d}f - udx - xd\cancel{u}$$
$$= udx + vdy - udx - xdu$$

$$= -x\,du + v\,dy$$

Reexpressing g as function of u, y
~~acc~~ accomplishes legendre transfor [illegible]

Examples from Thermodynamics

Int. En.: $dU = TdS - PdV$
$$T = \frac{\partial U}{\partial S} \qquad P = -\frac{\partial U}{\partial V}$$

Enthalpy: $H = U + P\cancel{V}$

$$dH = dU + VdP$$

$F = U - TS$ Helmholz Free energy
$G = H - TS$ Gibbs " "

$$H = p_i \dot{q}_i - L$$
$$dH = dp_i \dot{q}_i + p_i d\dot{q}_i - \left(\frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial L}{\partial t} dt\right)$$
$$= \dot{q}_i dp_i - \dot{p}_i dq_i - \frac{\partial L}{\partial t} dt$$

Start with the Lagrangian.

Use $p_i = \frac{\partial L}{\partial \dot{q}_i}$ to solve for

$$\dot{q}_i = \phi_i(p, q, t)$$

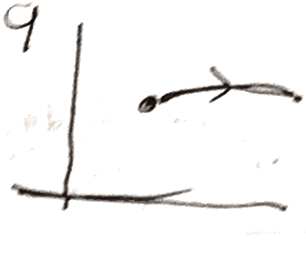
Replace in $p_i \dot{q}_i - L(q, \dot{q}, t)$

Get $H(p_i, q_i, t)$

$$\left.\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ -\dot{p}_i &= \frac{\partial H}{\partial q_i} \end{aligned}\right\} \rightarrow \text{Hamiltons' equations of motion}$$

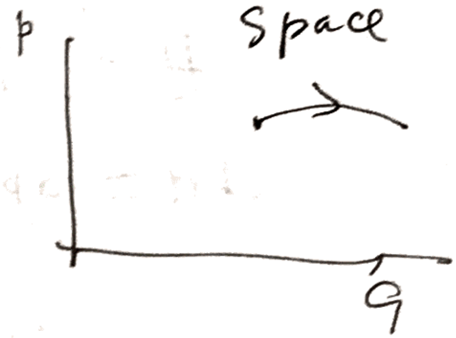
$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}$$

Space of "states"



$\Rightarrow$

Phase Space



Arbitrary

Evolution in phase space has interesting geometric properties. We will come to that later.

Examples:

1) Free particle:

$$L = \frac{m\dot{x}^2}{2}$$

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

$$\dot{x} = \frac{p}{m}$$

$$H = p\dot{x} - L = p\frac{p}{m} - \frac{m}{2}\left(\frac{p}{m}\right)^2$$

$$= \frac{p^2}{m} - \frac{p^2}{2m} = \frac{p^2}{2m}$$

$$\boxed{\begin{aligned}\dot{x} &= p/m \\ \dot{p} &= 0\end{aligned}}$$

2) Particle in a potential:

$$L = \frac{m\dot{x}^2}{2} - V(x)$$

$p = m\dot{x}$ $\quad \dot{x} = \frac{p}{m}$ (same)

$$H = p\dot{x} - L = \frac{p^2}{2m} + V(x)$$

$$\dot{x} = \frac{p}{m}$$

$$\dot{p} = -\frac{\partial}{\partial x} V(x) = -V'(x)$$

3) Particle in an electromagnetic field:

$$L = \frac{1}{2} m v^2 - q\phi + q\vec{A} \cdot \vec{v}$$

$$p_i = \frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i + qA_i$$

$$\dot{x}_i = (p_i - qA_i(x,t))/m$$

$$\vec{v} = (\vec{p} - q\vec{A})/m$$

$$H = \vec{p} \cdot \vec{v} - L = \vec{p} \cdot (\vec{p} - q\vec{A})/m - \frac{1}{2} m \left(\frac{p - qA}{m}\right)^2 + q\phi - q\vec{A} \cdot \left(\frac{p - qA}{m}\right)$$

$$= \frac{1}{2m}(\vec{p} - q\vec{A})^2 + q\phi$$

4) Polar coordinates:

$$T = \frac{mv^2}{2} = \frac{m}{2}\left(\dot r^2 + r^2 \sin^2\theta\, \dot\phi^2 + r^2 \dot\theta^2\right)$$

$$p_r = m\dot r \qquad p_\phi = m r^2 \sin^2\theta\, \dot\phi \qquad p_\theta = m r^2 \dot\theta$$

angular momenta

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\phi^2}{r^2 \sin^2\theta} + \frac{p_\theta^2}{r^2} \right)$$

5) $$L = L_0(q,t) + \dot q_i a_i(q,t) + \frac{1}{2} T_{ij}(q,t)\, \dot q_i \dot q_j$$

$$= L_0 + \underline{a} \cdot \dot{\underline{q}} + \frac{1}{2} \hat{\dot q}\, T \dot q$$

$$\underline{p} = \underline{a} + \underline{T}\dot{\underline{q}}$$

$$\dot{\underline{q}} = \underline{T}^{-1}(\underline{p} - \underline{a})$$

$$H = \hat{\dot{\underline{q}}}(\underline{p} - \underline{a}) - \frac{1}{2} \hat{\dot q}\, T \dot q - L_0$$

$$= \frac{1}{2}(\hat{\underline{p}} - \hat{\underline{a}})\, \underline{T}\, (\underline{p} - \underline{a}) - L_0$$

6) Spl. Relativity!

$$L = -m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}} - V(\underline{x})$$

(Expand and check!)

$$p_i = \frac{\partial L}{\partial v_i} = \frac{m_0 v_i}{\sqrt{1 - v^2/c^2}}$$

$$p_i v_i - L = \frac{m_0 v^2}{\sqrt{1 - \frac{v^2}{c^2}}} + m_0 c^2 \sqrt{1 - \frac{v^2}{c^2}} + V$$

$$= \frac{\cancel{m_0 v^2} + m_0 c^2 - \cancel{m_0 v^2}}{\sqrt{1 - v^2/c^2}} + V$$

$$= \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} + V$$

In general, $\eta_i = q_i$ $\eta_{i+n} = p_i$

$$\dot{\underline{\eta}} = \underline{J}\,\frac{\partial H}{\partial \underline{\eta}} = \underline{J}\,\underline{\nabla}_\eta H$$

$$\begin{pmatrix}\dot{\underline{q}}\\ \dot{\underline{p}}\end{pmatrix} = \underbrace{\begin{pmatrix}0 & +\mathbb{1}\\ -\mathbb{1} & 0\end{pmatrix}}_{J}\begin{pmatrix}\underline{\nabla}_q H\\ \underline{\nabla}_p H\end{pmatrix}$$

$\underline{J}$ in related to the symplectic form.

$$\tilde{\underline{J}}\,\underline{J} = \underline{\mathbb{1}} \qquad \tilde{\underline{J}} = -\underline{J}$$

$|J| = 1$.

Cyclic coordinates, Conservation theorems, Routh's procedure

Say $L = L(q_1, \ldots, q_{n-1}; \dot{q}_1, \ldots, \dot{q}_n, t)$

$\dot{p}_n = \dfrac{\partial L}{\partial q_n} = 0$ ∴ p_n is conserved

Note that $H = H(q_1, \ldots, q_{n-1}, p_1, \ldots p_{n-1}; \underset{\alpha}{\overset{p_n}{\updownarrow}}; t)$ and $\dot{q}_n = \dfrac{\partial H}{\partial \alpha}$

Corresponding velocity ~~momenta are conserved~~. could be determined after everything else is solved.

$L = L(q_1, \ldots, q_s; \dot{q}_1, \ldots \dot{q}_s, \dot{q}_{s+1}, \ldots, \dot{q}_n, t)$

Routh: Define Routhian

$$R(q_1, \ldots, q_s; \dot{q}_1, \ldots, \dot{q}_s, p_{s+1}, \ldots, p_n; t)$$

$$= \sum_{i=s+1}^{n} p_i \cdot \dot{q}_i - L$$

Only those degree of freedom that correspond to cyclic coordinates.

We have already been using Routh's procedure, effectively

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

θ cyclic

$$p_\theta = mr^2\dot{\theta}$$

$$R = p_\theta\dot{\theta} - L$$

$$= \frac{p_\theta^2}{2mr^2} - \frac{1}{2}m\dot{r}^2 + V(r)$$

$$= V_{eff}(r) - \frac{1}{2}m\dot{r}^2$$

Negative of an effective Lagrangian

Eqn of motion $\quad m\ddot{r} = -V_{eff}(r)$

$p_\theta = \ell \qquad \dot{\theta} = \frac{p_\theta}{mr^2}$

After solving for $r(t)$ we could solve for $\theta(t)$

Variational Principle

$$\delta I = \delta \int_{t_1}^{t_2} L\,dt$$

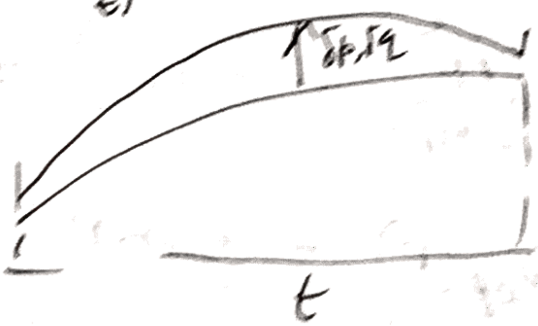
only δq var^n

$\delta p, \delta q$ var^n

$$\delta \int_{t_1}^{t_2} \left(p_i \dot{q}_i - H(p,q,t) \right) dt$$

$$= \int_{t_1}^{t_2} dt \left[\delta p_i \left\{ \dot{q}_i - \frac{\partial H}{\partial p_i} \right\} + p_i \delta \dot{q}_i - \frac{\partial H}{\partial q_i} \delta q_i \right]$$

$$= \int_{t_1}^{t_2} dt \left[\frac{d}{dt}(p_i \delta q_i) + \delta p_i \left\{ \dot{q}_i - \frac{\partial H}{\partial p_i} \right\} - \left(\dot{p}_i + \frac{\partial H}{\partial q_i} \right) \delta q_i \right]$$



$$= p_i \delta q_i \Big|_1^2 + \int \left[\delta p_i \left\{ \dot{q}_i - \frac{\partial H}{\partial p_i} \right\} - \left\{ \dot{p}_i + \frac{\partial H}{\partial q_i} \right\} \delta q_i \right] dt$$

If $\delta q_i(t_1) = 0 = \delta q_i(t_2)$, then

$$\delta \int (p_i \dot{q}_i - H)\,dt \Rightarrow$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

See that Modified Hamilton principle does not need $\delta p_i = 0$

However, if we let $\delta p_i, \delta q_i$ both be zero at the end,

$$\delta \int \left[p_i \dot{q}_i - H(p,q,t) - \frac{dF(p,q,t)}{dt} \right] dt$$

$= 0$

leads to equivalent results.

Using $F = p_i q_i$

$$\delta \int \left[- q_i \dot{p}_i - H \right] dt = 0$$

Gives the same equations.

Note that [redacted]

$$\int p_i \dot{q}_i \, dt = \int_C p_i \, dq_i \quad \text{(line integral)}$$

$$-\int q_i \dot{p}_i \, dt = -\int_C q_i \, dp_i$$

On a closed loop

$$\int_{C-C'} p_i \, dq_i = -\int_{C-C'} q_i \, dp_i$$

$$\delta \int p_i \, dq_i = \delta\left(-\int q_i \, dp_i\right)$$

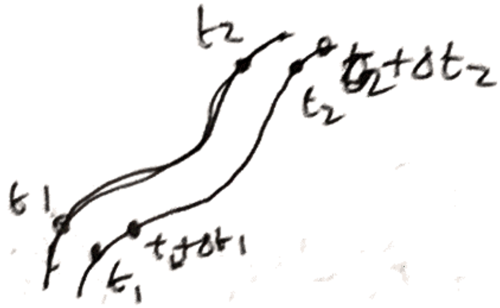
More about soon when we talk of differential forms

Principle of least action

Δ - variation

$$q_i(t,\alpha) = q_i(t,0) + \alpha\,\eta_i(t)$$

$L_\alpha = L(q_\alpha, \dot q_\alpha,$



$$\Delta \int_{t_1}^{t_2} L\,dt = \int_{t_1+\Delta t_1}^{t_2+\Delta t_2} L_\alpha\,dt - \int_{t_1}^{t_2} L_0\,dt$$

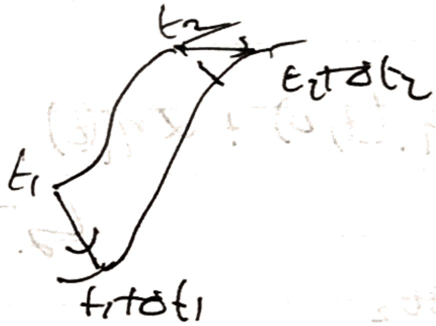
$$\Delta \int_{t_1}^{t_2} L\,dt = L(t_2)\Delta t_2 - L(t_1)\Delta t_1$$

$$+ \int_{t_1}^{t_2} \delta L\,dt$$

$$= L(t_2)\Delta t_2 - L(t_1)\Delta t_1$$

$$+ \frac{\partial L}{\partial \dot q_i}\delta q_i\Big|_1^2 + \int_1^2 \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot q_i}\right)\delta q_i\,dt$$

$$\Delta \int L\,dt = \left(L\Delta t + p_i \delta q_i\right)\Big|_1^2$$



$$\Delta q_i = \delta q_i + \dot{q}_i \Delta t$$

$$\Delta \int_1^2 L\,dt = \left[p_i \Delta q_i - (p_i \dot{q}_i - L)\Delta t\right]_1^2$$

$$= \left[p_i \Delta q_i - H\Delta t\right]_1^2$$

1) No time dependence
2) Only paths with H conserved
3) Set $\Delta q_i = 0$
rather that $\delta q_i = 0$

p_i satisfied $p_i = \frac{\partial L}{\partial \dot{q}_i}$

$$\Delta \int_1^2 L\,dt = -H(\delta t_2 - \delta t_1)$$

$$= -\Delta \int_1^2 H\,dt$$

$$\Delta \int_1^2 (L+H)\,dt = 0$$

$$\Delta \int p_i \dot{q}_i\,dt = 0$$

$$\Delta \int p_i\,dq_i = 0$$

Independent of time parametriz^n

$$\int p_i\,dq_i \rightarrow \text{action.}$$

One concrete example.

If $L = T - V(q)$

and $T = \frac{1}{2} M_{jk}(q)\dot{q}_j \dot{q}_k$

$p_i = M_{ij}\dot{q}_j \qquad \int p_i \dot{q}_i\,dt = 2\int_1^2 T\,dt$

The point is that $\Delta \int p_i \dot{q}_i dt = 0$ around 'the path' for fixed H paths. They include paths on which $p_i = \frac{\partial L}{\partial \dot{q}_i}$, with energy conservation satisfied.

If $L = T$, and T is constant, then we need to have $\Delta \int_1^2 dt = \delta(t_2 - t_2) = 0$

We are looking for the shortest time path for fixed kinetic energy. This is trivial for linear motion but non-trivial for rigid body rotation.

What to do when $L = T - V(q)$

$T = H - V(q)$, with H fixed.

$M_{ij} dq_i dq_j = d\rho^2 \leftarrow$ line element in q space

$$T = \frac{1}{2}\left(\frac{d\rho}{dt}\right)^2 = H - V(q)$$

If we just want to focus the path's geometry, not on time dependence

$$2\int T dt = \int \dot{\rho}\, d\rho$$

$$= \int \sqrt{2(H - V(q))}\, d\rho$$

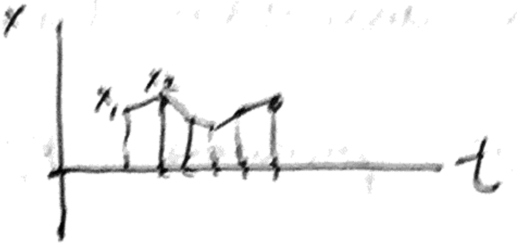
This is Jacobi's version of minimum/stationary action principle: very similar to shortest optical path (with position dependent refractive index). After finding $q(s)$, $t = \int \frac{d\rho}{\dot{\rho}}$

Aside on the discretized version of variational principle

Consider a free particle:

$$\int_1^2 L\,dt \xrightarrow{\text{discretized}} \sum_{i=1}^{N-1} \frac{m}{2}\left(\frac{x_{i+1}-x_i}{\Delta t}\right)^2 \Delta t$$

$N\Delta t = t_2 - t_1$



$$\text{Path} = \underset{\{x\}}{\operatorname{argmin}} \sum_{i=1}^{N-1} \frac{m}{2}\frac{(x_{i+1}-x_i)^2}{\Delta t}$$: Lagrangian

$$= \underset{\{x\}}{\operatorname{argmin}} \sum_{i=1}^{N-1}$$ [redacted] $$\max_{p_i} \left[p_i (x_{i+1}-x_i) - \frac{p_i^2}{2m}\Delta t \right]$$

$$= \underset{\{x\}}{\operatorname{argmin}}$$ [redacted] $$\max_{\{p\}} \sum_{i=1}^{N-1} \left[p_i \frac{(x_{i+1}-x_i)}{\Delta t} - \frac{p_i^2}{2m} \right] \Delta t$$

$$\hookrightarrow \int^2 (p\dot{q} - H)\,dt$$

Under some circumstances $\text{Path} = \underset{\{x\}}{\operatorname{argmin}} \sum \left[\frac{m}{2} \frac{(x_{i+1}-x_i)^2}{\Delta t^2} - V(x_i) \right] \Delta t$

[redacted]

In those cases

$$x_{path} = \underset{\{x\}}{\text{argmin}} \; \underset{\{p\}}{\max} \sum_i \left[p_i \frac{(x_{i+1}-x_i)}{\Delta t} - \frac{p_i^2}{2m} - V(x_i) \right] \Delta t$$

Note that now it is a min max problem over x and auxilliary variables p.

p_i maximizing produces $p_i = m \frac{\Delta x_i}{\Delta t}$

$= m \frac{(x_{i+1} - x_i)}{\Delta t}$

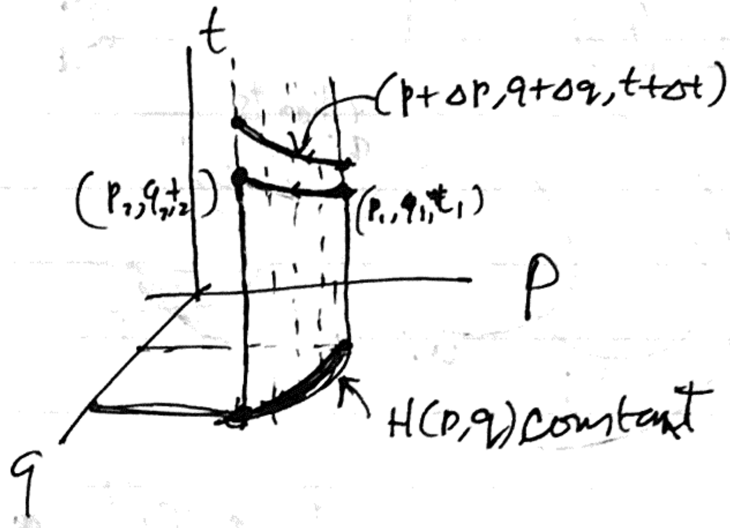
[redacted] Under certain conditions, [redacted] the order of $\underset{\{x\}}{\min} \; \underset{\{p\}}{\max}$ could be reversed

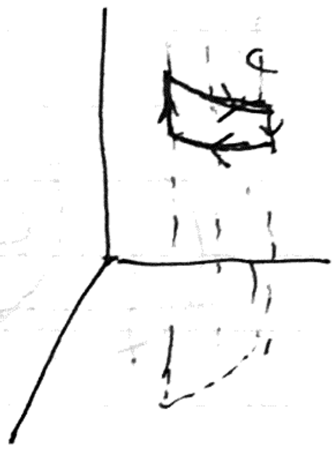
to $\underset{\{p\}}{\max} \; \underset{\{x\}}{\min}$. For us, we are doing

local variation conditions, and it does not matter as much.

The $2n+1$ dimensional pt. of view

Add t to p_i, q_i





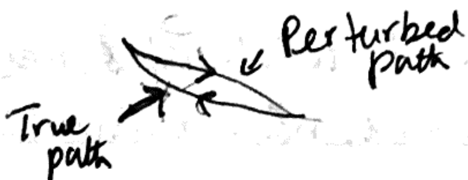
$$\Delta \int_1^2 p\,dq = -\oint_C (p\,dq - H\,dt)$$

because [redacted] $\oint_C H\,dt = H \oint_C dt = 0$

H constant surface

and on the vertical parts $\int p\,dq = 0$ [redacted] since q does not change

Hamilton's variational principle could be written as



$$\oint_C (p_i\,dq_i - H\,dt) = 0$$