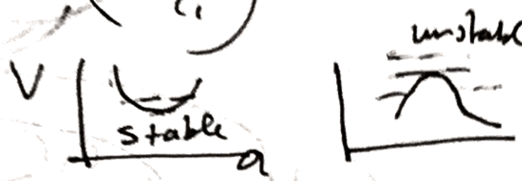
Oscillations

Time invariant system

Equilibrium $\quad Q_i = -\left(\frac{\partial V}{\partial q_i}\right) = 0$

$\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = 0$



Stable equilibrium: small perturbation produces small bounded motion around the rest position.

Motion in the visinity of a stable equilibrium.

$$q_i = q_{0i} + \eta_i$$

$$V(q_1, \ldots, q_n) = V(q_{01} + \eta_1, \ldots, q_{0n} + \eta_n)$$

$$= V(q_{01}, \ldots, q_{0n}) + \left(\frac{\partial V}{\partial q_i}\right)_0 \eta_i + \frac{1}{2}\left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_0 \eta_i \eta_j + \cdots$$

[summation convention on]

$\left(\frac{\partial V}{\partial q_i}\right)_0 = 0 \quad$ if $q_{0i} \to$ eqm

Drop the constant and stop at the lead term

$$V = \frac{1}{2}\left(\frac{\partial^2 V}{\partial q_i \partial q_j}\right)_0 \eta_i \eta_j$$

$$= \frac{1}{2} V_{ij} \eta_i \eta_j$$

V_{ij} is a constant symmetric matrix.

In time invariant systems,

$$T = \frac{1}{2} m_{ij}(q_1, \dots, q_n)\, \dot{q}_i \dot{q}_j$$

$$m_{ij}(q_1, \dots, q_n) = m_{ij}(q_{01}, \dots, q_{0n}) + \left(\frac{\partial m_{ij}}{\partial q_k}\right)_0 \eta_k + \dots$$

$$\dot{q}_i = \dot{\eta}_i$$

T has terms like $\frac{1}{2}(m_{ij})_0 \dot{\eta}_i \dot{\eta}_j + O(\eta \dot{\eta} \dot{\eta})$

ignore

we want to keep only second order terms in η_i. $(m_{ij})_0 \to T_{ij}$, a constant symmetric matrix, again

$$T = \frac{1}{2} T_{ij} \dot{\eta}_i \dot{\eta}_j$$

$$L = \frac{1}{2}\left(T_{ij}\dot{\eta}_i\dot{\eta}_j - V_{ij}\eta_i\eta_j\right)$$

Equation of motion

$$T_{ij}\ddot{\eta}_j + V_{ij}\eta_j = 0$$

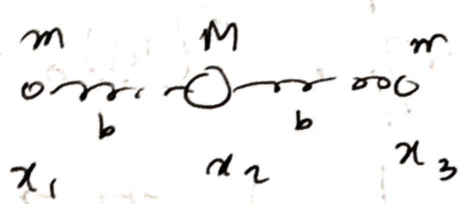
[redacted] In many problem

$T_{ij} = T_i\delta_{ij}$ no sum over i.

$T_i\ddot{\eta}_i + V_{ij}\eta_j = 0$, " " " "

Before we go on, let us look at an example that we will return to soon.

Linear triatomic molecule with symmetry



$$T = \frac{1}{2}(m\dot{x}_1^2 + M\dot{x}_2^2 + m\dot{x}_3^2)$$

$$V = \frac{k}{2}(x_2 - x_1 - b)^2 + \frac{k}{2}(x_3 - x_2 - b)^2$$

$$\eta_i = x_i - x_{0i}$$

At equilibrium

$$x_{02} - x_{01} = b = x_{03} - x_{02}$$

$$V = \frac{k}{2}\left\{(\eta_2 - \eta_1)^2 + (\eta_3 - \eta_2)^2\right\}$$

$$= \frac{k}{2}\left[\eta_1^2 + 2\eta_2^2 + \eta_3^2 - 2\eta_1\eta_2 - 2\eta_2\eta_3\right]$$

$$V_{ij} \to \underline{V} = \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix}$$

$$T = \frac{1}{2}(m\dot{\eta}_1^2 + M\dot{\eta}_2^2 + m\dot{\eta}_3^2)$$

$$T_{ij} \to \underline{T} = \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix}$$

Back to generalities:
Usually V_{ij} (and sometimes T_{ij}) end up coupling all coordinates.
How do we solve it?

Try oscillatory solutions of the form:

$$\eta_i = C a_i e^{i\omega t}$$

$$T_{ij} \ddot{\eta}_i + V_{ij} \eta_j = 0$$

$$\Rightarrow \quad T_{ij}(-\omega^2 C a_j) + V_{ij}(C a_j) = 0$$

or $(V_{ij} - \omega^2 T_{ij}) a_j = 0$

Matrix version

$$(\underline{V}_a - \lambda \underline{I}) \underline{a} = 0$$

$\omega^2 = \lambda$

Generalized eigenvector problem

$$|\underline{V} - \lambda \underline{T}| = 0$$

allows for non-trivial $\underline{a}$

If T was identity λ's would be real and a_k's orthogonal.

$$\underline{V}\,\underline{a}_k = \lambda_k \underline{T}\,\underline{a}_k$$

$$\underline{a}_l^+ \underline{V} = \lambda_l^* \underline{a}_l^+ \qquad a_k^+ = \tilde{a}_k^*$$

$$\underline{a}_l^+ \underline{V}\,\underline{a}_k = \lambda_k a_l^+ T a_k$$

$$= \lambda_l^* a_l^+ T a_k$$

If $(\lambda_k - \lambda_l^*) a_l^+ T a_k = 0$

$l = k$ $\quad a_k^+ T a_k$ is real

Physical K.E. is also positive definite so $a_k \neq a$

$\Rightarrow$ $a_k^+ T a_k > 0$

$\Rightarrow$ $\lambda_k = \lambda_k^*$ $\Rightarrow$ λ_k are real

Then $(\lambda_k - \lambda_l^*)\, \underline{a}_l^{\dagger} T \underline{a}_k = 0$

is the same as $(\lambda_k - \lambda_l)\, \underline{a}_l^{\dagger} T \underline{a}_k = 0$

If $\lambda_k \neq \lambda_l$ $\quad \underline{a}_l^{\dagger} T \underline{a}_k = 0$

[When two roots are the same but $a_k \neq a_l$, needs more careful discussion.]

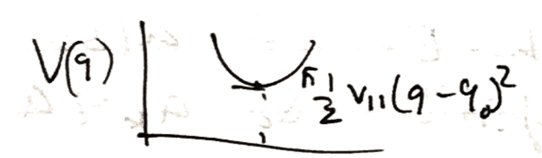
Also

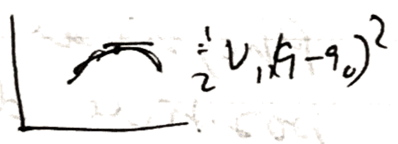
$$\lambda_k = \frac{\underline{a}_k^{\dagger} V \underline{a}_k}{\underline{a}_k^{\dagger} T \underline{a}_k}$$

If V is positive semidefinite ($\equiv$ Stability)

$\underline{a}_k^{\dagger} V a_k \geq 0$

$\Rightarrow \lambda_k \geq 0$





Stable $V_{11} > 0$ $\qquad V_{11} < 0$

$V = [V_{11}]$

[Note on $a_k^+ T a_k$ etc.

$\underline{a}_k$ is complex in general

$\underline{a}_k = \underline{\alpha}_k + i\underline{\beta}_k$ $\quad$ α_k, β_k real and imag part

$$a_k^+ T a_k = \underline{\tilde{\alpha}}_k \underline{T} \alpha_k + (-i\tilde{\beta}_k) T \beta_k$$
$$+ i \{ \underline{\tilde{\alpha}}_k \underline{T} \underline{\beta}_k - \underline{\tilde{\beta}}_k \underline{T} \underline{\alpha}_k \}$$

Transpose of each other for a scalar.

$$a_k^+ T a_k = \tilde{\alpha}_k T \alpha_k + \tilde{\beta}_k T \beta_k$$

Think of α_k & β_k as velocity patterns. If any of them are non zero, physical K.E. should be non zero.]

We could scale $\underline{a}_k$ so that

$$a_k^+ \, \underline{T} \, a_k = 1$$

Organize $\underline{A} = [\underline{a}_1 \cdots \underline{a}_n]$

column vectors

$$\underline{A}^+ \, \underline{T} \, \underline{A} = \underline{\mathbb{1}}$$

identity matrix.

Since the eigenvalues and the matrix are real, it turns out a_k are real.

So $\quad \underline{\tilde{A}} \, \underline{T} \, \underline{A} = \underline{\mathbb{1}}$

Similarity transformation $\quad C' = BCB^{-1}$

Congruence $\quad$ " $\quad C' = \tilde{A} C A$

For orthogonal matrices A, they are ~~both~~. the same, i.e. $C' = \tilde{A} C A$ is ~~both~~ Congruence and similarity transformation. In our case, in general, A is not orthogonal.

Let us introduce a diagonal matrix

$\underline{\lambda} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$ or $\lambda_{\ell k} = \lambda_k \delta_{\ell k}$ (no sum over k)

$$V_{ij} a_{jk} = T_{ij} a_{jk} \lambda_k$$

(no sum over k)

can now be written as

$$V_{ij} a_{jk} = T_{ij} a_{j\ell} \lambda_{\ell k}$$

Or $\underline{V}\,\underline{A} = \underline{T}\,\underline{A}\,\underline{\lambda}$

If. $\underline{\tilde{A}}\,\underline{T}\,\underline{A} = \underline{\underline{1}}$

then $\underline{\tilde{A}}\,\underline{V}\,\underline{A} = \underline{\lambda}$

Nate Hunt

So, we want to find a congruence tranformation that reduces $\underline{T}$ to identity matrix and $\underline{V}$ to a diagonal matrix. Simultaneous diagonalization of two quadratic forms!

Here's the algorithm.

$$T = \frac{1}{2}\dot{\tilde{\underline{\eta}}}\,\underline{T}\,\dot{\underline{\eta}} \qquad V = \frac{1}{2}\tilde{\underline{\eta}}\,\underline{V}\,\underline{\eta}$$

Diagonalize $\underline{T}$ by finding orthogonal matrix O_1 and tranformed variables $\underline{y}$

$$\underline{\eta} = \underline{O}_1\,\underline{y}. \qquad T = \frac{1}{2}\dot{\tilde{\underline{y}}}\,\tilde{O}_1\,\underline{T}\,O_1\,\dot{\underline{y}}$$

$$V = \frac{1}{2}\tilde{\underline{y}}\,\tilde{O}_1\,V\,O_1\,\underline{y}$$

Let $\tilde{\underline{O}}_1\,\underline{T}\,\underline{O}_1 = \begin{pmatrix} m_1 & & & \\ & m_2 & & \\ & & \ddots & \\ & & & m_n \end{pmatrix}$

$m_i > 0$

Choose $\underline{S} = \begin{pmatrix} 1/\sqrt{m_1} & & & \\ & 1/\sqrt{m_2} & & \\ & & \ddots & \\ & & & 1/\sqrt{m_n} \end{pmatrix}$

and let $\underline{y} = \underline{S}\,\underline{\xi}$

Note that $\tilde{\underline{S}}\,\tilde{\underline{O}}_1 \underline{T}\, \underline{O}_1 \underline{S} = \mathbb{1}$

[Remember $\tilde{S} = S$]

So $\quad T = \frac{1}{2} \tilde{\dot{\underline{\xi}}}\, \tilde{\underline{S}}\, \tilde{\underline{O}}_1 \underline{T}\, \underline{O}_1 \underline{S}\, \dot{\underline{\xi}}$

$\quad = \frac{1}{2} \tilde{\dot{\underline{\xi}}}\, \dot{\underline{\xi}} = \frac{1}{2} \sum_i \dot{\xi}_i^2$

$$V = \frac{1}{2} \tilde{\underline{\xi}}\, \underbrace{\tilde{\underline{S}}\, \tilde{\underline{O}}_1 \underline{V}\, \underline{O}_1 \underline{S}}\, \underline{\xi}$$

$$= \frac{1}{2} \tilde{\underline{\xi}}\, \underline{B}\, \underline{\xi}$$

$\underline{B}$ is a symetric matrix. Diagonalise it with an orthogonal matrix $\underline{O}_2$.

In other words, let $\underline{\xi} = \underline{O}_2 \underline{\zeta}$

$$V = \frac{1}{2} \tilde{\underline{\zeta}}\, \underbrace{\tilde{\underline{O}}_2 \underline{B}\, \underline{O}_2}_{\text{diagonal } \underline{\lambda}}\, \underline{\zeta} = \frac{1}{2} \tilde{\underline{\zeta}}\, \underline{\lambda}\, \underline{\zeta}$$

Note that $\quad T = \frac{1}{2} \tilde{\dot{\underline{\zeta}}}\, \underbrace{\tilde{\underline{O}}_2 \underline{O}_2}_{\mathbb{1}}\, \dot{\underline{\zeta}} = \frac{1}{2} \tilde{\dot{\underline{\zeta}}}\, \dot{\underline{\zeta}}$

So $\underline{\lambda} = \underline{\tilde{O}}_2 \underline{\tilde{S}} \underline{\tilde{O}}_1 \underline{V} \underline{O}_1 \underline{S} \underline{O}_2$

Our $\underline{A} = \underset{\text{orthogonal}}{\underline{O}_1}\,\underset{\text{diagonal}}{\underline{S}}\,\underset{\text{orthogonal}}{\underline{O}_2}$

At the end: $\mathcal{L} = \frac{1}{2} \underline{\dot{\tilde{\xi}}} \underline{\dot{\xi}} - \frac{1}{2} \underline{\tilde{\xi}} \underline{\lambda} \underline{\xi}$

$$= \frac{1}{2} \sum_i \dot{\xi}_i^2 - \frac{1}{2} \sum_i \lambda_i \xi_i^2$$

For stable potential $\lambda_i = \omega_i^2 \geq 0$

$$\mathcal{L} = \underbrace{\sum_i \left(\frac{1}{2} \dot{\xi}_i^2 - \frac{1}{2} \omega_i^2 \xi_i^2 \right)}_{\text{independent oscillators}}$$

ξ_i Normal modes/coordinates

ω_i Frequencies of free vibration OR Resonant frequencies

Two examples :

1. $T = \frac{1}{2} m(\dot{x}_1^2 + \dot{x}_2^2)$

$V = \frac{1}{2} V_{ij} x_i x_j$

$\underline{T} = \begin{pmatrix} m & \\ & m \end{pmatrix}$ No need to diagonalize.

$\underline{V} = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}$ $V_{21} = V_{12}$

Diagonalize V

$$\begin{vmatrix} V_{11} - \lambda & V_{12} \\ V_{21} & V_{22} - \lambda \end{vmatrix} = 0$$

$$(\lambda - V_{11})(\lambda - V_{22}) - V_{12} V_{21} = 0$$

$$\lambda^2 - (V_{11} + V_{22})\lambda + (V_{11} V_{22} - V_{12} V_{21}) = 0$$

$$\lambda_{1,2} = \frac{1}{2}\left(V_{11}+V_{22} \pm \sqrt{(V_{11}+V_{22})^2 - 4(V_{11}V_{22}-V_{12}V_{21})}\right)$$

$$= \frac{1}{2}\left(V_{11}+V_{22} \pm \sqrt{(V_{11}-V_{22})^2 + 4V_{12}V_{21}}\right)$$

$$= \frac{1}{2}\left(V_{12}+V_{22} \pm \sqrt{(V_{11}-V_{22})^2 + 4V_{12}^2}\right)$$

Two interesting cases:

a) $|V_{12}| \ll |V_{11}-V_{22}|$

Let $\frac{V_{12}}{V_{11}-V_{22}} = \delta$

$$\lambda_{1,2} = \frac{1}{2}\left[V_{11}+V_{22} \pm (V_{11}-V_{22})(1+4\delta^2)^{1/2}\right]$$

$$\lambda_1 \simeq V_{11} + (V_{11}-V_{22})\cdot\delta^2 = V_{11}+V_{12}\delta$$

$$\lambda_2 \simeq V_{22} - (V_{11}-V_{22})\,\delta^2 = V_{22}-V_{12}\delta$$

~~The eigenvectors are~~

$$\underline{V} - \lambda_1 \underline{\mathbb{I}} = \begin{pmatrix} -V_{12}\delta & V_{12} \\ V_{12} & V_{22} - V_{11} + V_{12}\delta \end{pmatrix} = \begin{pmatrix} -V_{12}\delta & V_{12} \\ (V_{11}-V_{22})\delta & -(V_{11}-V_{22}) \times (1+\delta^2) \end{pmatrix}$$

$$(\underline{V} - \lambda_1 \underline{\mathbb{I}})\underline{a}_1 = 0$$

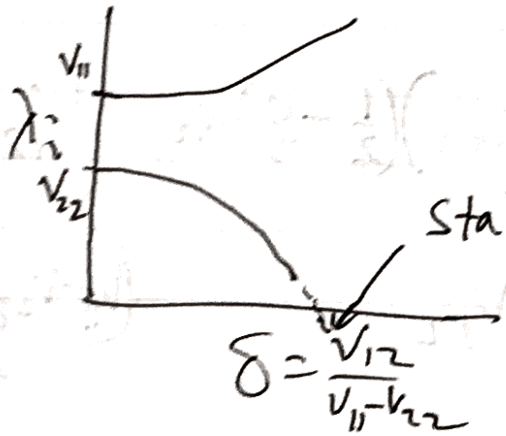
When $V_{12} = 0$, this is $\begin{pmatrix} 0 & 0 \\ 0 & V_{22} - V_{11} \end{pmatrix}$

has eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Now $\underline{a}_1 \simeq \begin{pmatrix} 1 \\ \delta \end{pmatrix}$ upto order δ^2

Similar analysis shows $a_2 \simeq \begin{pmatrix} -\delta \\ 1 \end{pmatrix}$

Small mixing of modes and 'level repulsion'



b) $|V_{12}| \gg |V_{11} - V_{22}|$

$$\varepsilon = \frac{V_{11} - V_{22}}{8 V_{12}}$$

$$\lambda_1, \lambda_2 = \frac{V_{11} + V_{12}}{2} \pm |V_{12}| \left[1 + (4\varepsilon)^2\right]^{1/2}$$

$$= \frac{V_{11} + V_{12}}{2} \pm V_{12} \left(1 + 16\varepsilon^2\right)^{1/2}$$

(Assume $V_{12} > 0$)

$$\lambda_1 \approx \frac{1}{2}(V_{11} + V_{22}) + V_{12} + 8\varepsilon^2 V_{12}$$

$$= \frac{1}{2}(V_{11} + V_{22}) + V_{12} + (V_{11} - V_{12})\varepsilon$$

Similarly

$$\lambda_2 = \frac{1}{2}(V_{11} + V_{22}) - V_{12} - (V_{11} - V_{12})\varepsilon$$

$$\underline{V} - \lambda_1 \underline{\mathbb{1}} = \begin{pmatrix} (V_{11} - V_{22})\left(\frac{1}{2} - \varepsilon\right) - V_{12} & V_{12} \\ V_{12} & -(V_{11} - V_{22})\left(\frac{1}{2} + \varepsilon\right) - V_{12} \end{pmatrix}$$

$$= \cancel{(V_{11} = V_{22})} \; V_{12} \begin{pmatrix} -1 + 4\varepsilon + O(\varepsilon^2) & 1 \\ 1 & -1 - 4\varepsilon \end{pmatrix}$$

$$(\underline{V} - \lambda_1 \underline{\mathbb{1}}) \, \underline{a}_1 = 0$$

If $\varepsilon = 0$ $\quad (V - \lambda_1 \mathbb{1}) = V_{12} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$

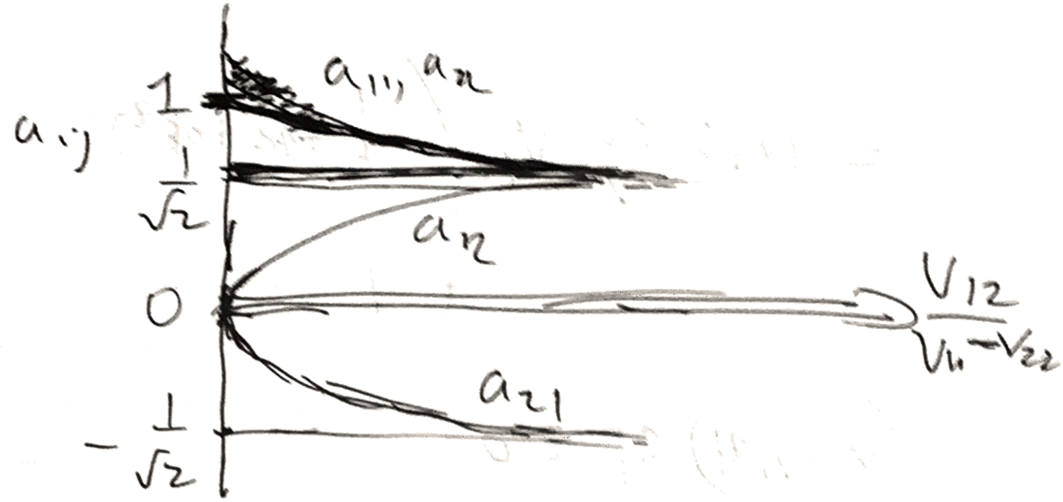
$$\underline{a}_1 \to \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Now $\begin{pmatrix} 1 + 2\varepsilon + O(\varepsilon^2) \\ 1 - 2\varepsilon + O(\varepsilon^2) \end{pmatrix}$

~~With~~ Similarly $\underline{a}_2 \to \begin{pmatrix} -(1-2\varepsilon) \\ (1+2\varepsilon) \end{pmatrix}$

With normalization

$$\underline{A} = \begin{bmatrix} \frac{1}{\sqrt{2}}(1+2\varepsilon) & -\frac{1}{\sqrt{2}}(1-2\varepsilon) \\ \frac{1}{\sqrt{2}}(1-2\varepsilon) & \frac{1}{\sqrt{2}}(1+2\varepsilon) \end{bmatrix}$$



Before we move on, let us consider the case of degenerate roots.

Let $\lambda_k = \lambda_l$ and a_k' and a_l' are the corresponding eigenvectors.

Let's say $\tilde{a}_l' T a_k' = \tau \neq 0$. Consider

$$a_l = c_1 a_k' + c_2 a_l'$$

$$\tilde{a}_l T a_k' = c_1 \tilde{a}_k' T a_k' + c_2 \tilde{a}_l' T a_k'$$

Say $\tilde{a}_k' T a_k = 1$

Choose $\frac{c_1}{c_2} = - \tilde{a}_l' T a_k' = -\tau$

$$\tilde{a}_l T a_k' = 0$$

Like Gram-schmidt orthogonalization.

For higher multiplicity $\quad a_2 = \text{contr of } a_1', a_2'$

$\qquad\qquad\qquad\qquad\quad\; a_3 = \;\text{"}\;\;\text{"}\;\; a_1', a_2, a_3'$

$\qquad\qquad\qquad\qquad\quad\;\; \vdots$

Example: $\begin{bmatrix} V_{11} & V_{12} \\ V_{12} & V_{22} \end{bmatrix} \qquad T = \begin{pmatrix} m & \\ & m \end{pmatrix}$

If $\begin{bmatrix} V_{11} & 0 \\ 0 & V_{11} \end{bmatrix}$ has a double root $\lambda_{1,2} = V_{11}$

If $V_{12} \to 0$ first and the $V_{11} \to V_{22}$

we get $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

If $V_{11} \to V_{22}$ first $V_{12} \to 0$ later

$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$

Arbitraryness of choice.

Linear triatomic Molecule

O = C = O

$$\eta_{i+1} - \eta_i = x_{i+1} - x_i - b$$

$$T = \frac{m}{2}\left(\dot{\eta}_1^2 + \dot{\eta}_3^2\right) + \frac{M}{2}\dot{\eta}_2^2$$

$$\underline{T} = \begin{pmatrix} m & & \\ & M & \\ & & m \end{pmatrix}$$

$$V = \frac{k}{2}\left(\eta_1^2 + 2\eta_2^2 + \eta_3^2 - 2\eta_1\eta_2 - \eta_2\eta_3\right)$$

$$\underline{V} = \begin{pmatrix} k & -k & \\ -k & 2k & -k \\ & -k & k \end{pmatrix}$$

Try sol$^{\underline{n}}$ with symm $\underline{V}\,\underline{a} = \omega^2 \underline{T}\,\underline{a}$ $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ $\begin{pmatrix} 1 \\ -a \\ 1 \end{pmatrix}$

Blind way forward:

$$|V-\omega^2 T| = \begin{vmatrix} k-\omega^2 m & -k & \\ -k & 2k-\omega^2 M & -k \\ & -k & k-\omega^2 m \end{vmatrix}$$

$$= (k-\omega^2 m)\begin{vmatrix} 2k-\omega^2 M & -k \\ -k & k-\omega^2 m \end{vmatrix}$$

$$+ k \begin{vmatrix} -k & -k \\ 0 & k-\omega^2 m \end{vmatrix}$$

$$= (k-\omega^2 m)\left[(2k-\omega^2 M)(k-\omega^2 m) - k^2\right]$$

$$-k^2(k-\omega^2 m)$$ [redacted]

$$= (k-\omega^2 m)\left[(2k-\omega^2 M)(k-\omega^2 m) - 2k^2\right]$$

$$= (k-\omega^2 m)\left[\omega^4 mM - \omega^2(2m+M)k\right]$$

$$= \omega^2(k-\omega^2 m)\left[k(M+2m) - \omega^2 Mm\right]$$

So $\omega^2(k-\omega^2 m)\left[k(M+2m) - \omega^2 Mm\right] = 0$

$$\omega^2 = 0, \quad \frac{k}{m}, \quad \frac{k(M+2m)}{Mm}$$

First mode with $\omega = 0$

$\Rightarrow \ddot{\xi}_1 = 0$ That is just rigid translation of the system: C.M. moving with relative positions fixed.

$$\begin{pmatrix} (k-\omega_j^2 m) & -k & \\ -k & 2k-\omega_j^2 M & -k \\ & -k & k-\omega_j^2 m \end{pmatrix} \begin{pmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \end{pmatrix} = 0$$

$\omega_1 = 0$ allows $a_{11} = a_{21} = a_{31}$

as a solution.

With $\tilde{a} T a = 1$ $\quad m(a_{1j}^2 + a_{3j}^2) + M a_{2j}^2 = 1$

$$\begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} = \frac{1}{\sqrt{2m+M}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\omega_2^2 = \frac{k}{m} \Rightarrow \begin{pmatrix} 0 & -k & 0 \\ -k & 2k - \frac{kM}{m} & -k \\ 0 & -k & 0 \end{pmatrix} \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} = 0$$

$a_{22} = 0 \qquad a_{12} = -a_{32}$

$$\begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} = \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

The center atom is a rest and the two others are going in opposite directions.

$$\omega_3^2 = k\left(\frac{M+2m}{mM}\right)$$

$$\begin{pmatrix} -k\frac{2m}{M} & -k & \\ -k & -k\frac{M}{m} & k \\ & -k & -k\frac{2m}{M} \end{pmatrix}$$

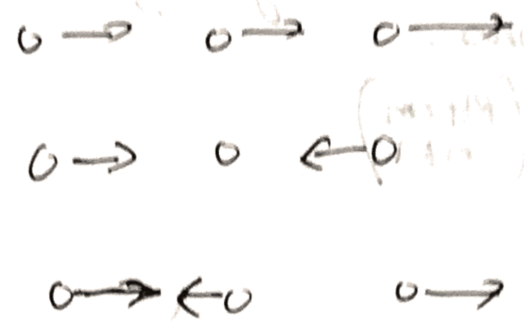
Seems $a_{13} : a_{23} : a_{33}$

$= 1 : -\frac{2m}{M} : 1.$

will work. Normalizing

$$\begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} = \frac{1}{\sqrt{2m\left(1+\frac{2m}{M}\right)}} \begin{pmatrix} 1 \\ -\frac{2m}{M} \\ 1 \end{pmatrix}$$

The three modes are



$$\begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2m+M}} & \frac{1}{\sqrt{2m}} & \frac{1}{\sqrt{2m(1+\frac{2m}{M})}} \\ \frac{1}{\sqrt{2m+M}} & 0 & -\frac{1}{\sqrt{2M(\frac{M}{m}+2)}} \\ \frac{1}{\sqrt{2m+M}} & -\frac{1}{\sqrt{2m}} & \frac{1}{\sqrt{2m(1+\frac{2m}{M})}} \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix}$$

$$\zeta_1 = \frac{m(\eta_1+\eta_3) + M\eta_2}{\sqrt{2m+M}}$$

$$\zeta_2 = \sqrt{\frac{m}{2}}(\eta_1 - \eta_3)$$

$$\zeta_3 = \sqrt{\frac{m}{1+\frac{2m}{M}}}(\eta_1+\eta_3-2\eta_2)$$

Forced vibrations and effect of dissipation

[redacted]

$$T_{ij}\ddot{\eta}_j + V_{ij}\eta_j = F_j$$

external force

$$\underline{T}\,\ddot{\underline{\eta}} + \underline{V}\,\underline{\eta} = \underline{F}$$

$$TA\ddot{\xi} + VA\xi = F$$

multiply by $\hat{A}$ $\quad \hat{A}F = Q$

$$\hat{A}\underline{T}\underline{A}\ddot{\underline{\xi}} + \hat{\underline{A}}\underline{V}\underline{A}\underline{\xi} = \underline{Q}$$

$$\ddot{\underline{\xi}} + \underline{\Lambda}\,\underline{\xi} = \underline{Q}$$

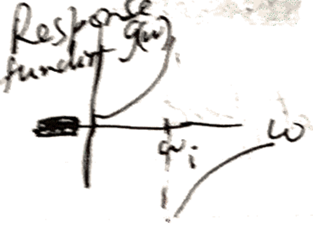
Component by component

$$\ddot{\xi}_i + \lambda_i \xi_i = Q_i$$

or $\ddot{\xi}_i + \omega_i^2 \xi_i = Q_i$

Let $Q_i(t) = Q_{0i}\cos(\omega t + \delta_i)$

Try [redacted] $\xi_i = B_i \cos(\omega t + \delta_i)$



$$B_i = \frac{Q_{0i}}{\omega_i^2 - \omega^2}$$

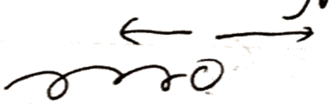
$$\eta_j = a_{ji}\,\xi_i = \frac{a_{ji}\,Q_{0i}\cos(\omega t + \delta_i)}{\omega_i^2 - \omega^2}$$

1) First [redacted] $Q_{0i} \neq 0$

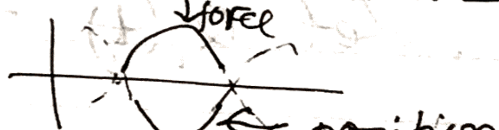
2) Also $\omega^2 \to \omega_i^2$ produces large response (Resonance!)

3) Phase change as ω [redacted] crosses ω_i^2.

Push slowly (ω low) and the particle moves with force



Jiggle fast and acceleration [redacted] matches force



Effect of [redacted] dissipation

If we have $\ddot{\xi}_j + F_j \dot{\xi}_j + \omega_j^2 \xi_j = 0$

drag coeff

$$\xi_j = c_j e^{-i\omega_j' t}$$

$$\omega_j'^2 + i\omega_j' F_j - \omega_j^2 = 0$$

$$\omega_j' = \pm\sqrt{\omega_j^2 - \frac{F_j^2}{4}} - i\frac{F_j}{2}$$

damping rate

$$\xi_j = C_j e^{-i\omega_j' t} = C_j e^{\mp i\sqrt{\omega_j^2 - \frac{F_j^2}{4}}\, t} e^{-F_j t/2}$$

$$= c_j e^{\mp 2\pi i \nu_j t - \varkappa t}$$

In general $\eta_j = c a_j e^{-i\omega t}$

General Rayleigh Dissipation function $\mathcal{F} = \frac{1}{2} F_{ij} \dot{\eta}_i \dot{\eta}_j$

$V_{jk} a_k - i\omega F_{[redacted]k} a_k - \omega^2 T_{jk} a_k = 0$

$$(\underline{V} + \gamma \underline{F} + \gamma^2 \underline{T})\,\underline{a} = 0$$

$\gamma = -i\omega = -\varkappa - 2\pi i \nu$

Note that as γ, ... in a soln γ^*, a^* in a solution too

Also note that [redacted] $a^\dagger V a + \gamma a^\dagger F a + \gamma^2 a^\dagger T a = 0$ is a real quadratic equation. If γ is complex

$$2\,\mathrm{Re}(\gamma) = \gamma + \gamma^* = -\frac{a^\dagger F a}{a^\dagger T a} < 0 \Rightarrow \kappa > 0.$$

For forced motion

$$V_{jk}\,\eta_k + F_{jk}\,\dot{\eta}_k + T_{jk}\,\ddot{\eta}_k = F_{0j}\,e^{-i\omega t}$$

$$\eta_j = A_j e^{-i\omega t}$$

$$(\underline{V} - i\omega \underline{F} - \omega^2 \underline{T})\underline{A} = F_0$$

$$A = (\qquad)^{-1} F_0$$

$$A_j = \frac{D_j(\omega)}{D(\omega)}$$ [redacted]

[redacted] using Cramer's rule

Ask when $D(\omega)$ gets small (resonance) [redacted]

$$D(\omega) = G \prod_j (\omega - \omega_j)(\omega + \omega_j^*)$$

[redacted] Using $\omega_j = 2\pi\nu_j - i\kappa_j$ and $\omega = 2\pi\nu$ [redacted]

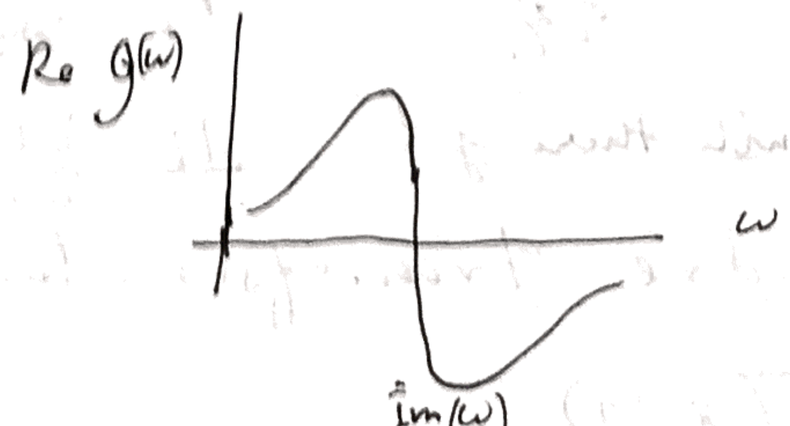
$$D(\omega) D(\omega)^* = \prod_j \left(4\pi^2(\nu - \nu_j)^2 + \kappa_j^2\right)\left(4\pi^2(\nu + \nu_j)^2 + \kappa_j^2\right)$$

So as ν passes ω_j, and as long as $\kappa_j \ll |\nu_j - \nu_{j'}|$, we will see a resonant peak.

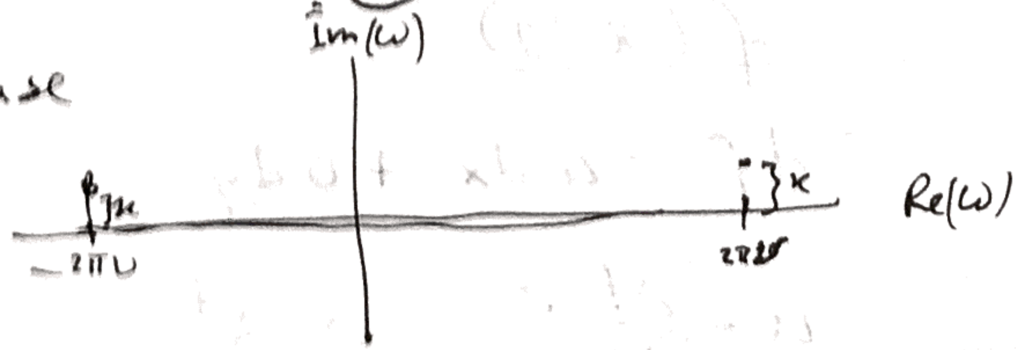
One variable [redacted] case

$$(\omega_0^2 - i\omega\mathcal{F} - \omega^2)A = F_0$$

$$A = \frac{1}{\omega_0^2 - \omega^2 - i\omega\mathcal{F}} F_0 = g(\omega) F_0$$



1D case



General