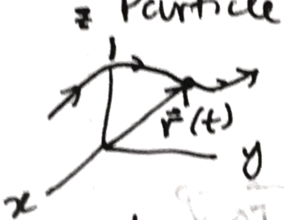
Notations and Preliminary Discussions

Particle motion



$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{\vec{r}} \, , \quad \vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

Linear momentum: $\vec{p} = m\vec{v}$

Newton 2: $\vec{F} = \frac{d\vec{p}}{dt} = \dot{\vec{p}}$

[Newton 1: $\vec{F} = 0$, m const, $\Rightarrow$ $\dot{\vec{v}} = 0$]

Momentum turns out to be something more fundmental than just being a concept derived from velocity and mass: think of particles which could change into other particles.

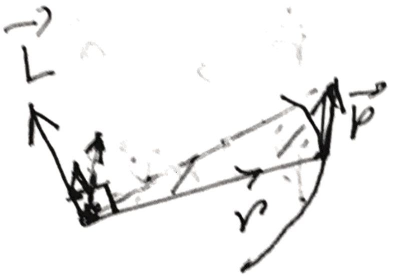
<u>Conservation</u> of <u>Linear momentum</u>

Total force zero $\Rightarrow$ Linear momentum conserved

$\vec{F} = 0$ $\Rightarrow$ p constant

Angular Momentum

$$\vec{L} = \vec{r} \times \vec{p}$$



$$\vec{l} = \vec{r} \times \vec{v}$$
$$= m(\vec{r} \times \vec{v})$$

Relation to Kepler 2: $\frac{1}{2} \vec{r} \times \vec{v}$ = area swept at origin.

Note that
$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p})$$
$$= \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$
$$= \underbrace{\vec{v} \times (m\vec{v})}_{0} + \vec{r} \times \vec{F}.$$

$\vec{N} = \vec{r} \times \vec{F}$ is the [redacted] torque.

$$\frac{d\vec{L}}{dt} = \vec{N}$$

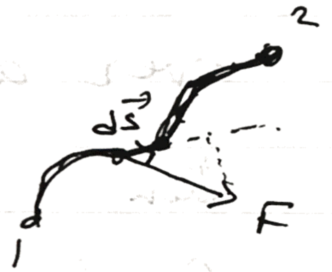
Conservation of Angular Momentum

Total torque zero $\Rightarrow$ Angular Mom. conserved
$\vec{N} = 0 \Rightarrow \vec{L}$ constant !

Ex. Central force

Work done by external force $\vec{F}$

$$W_{12} = \int_1^2 \vec{F} \cdot d\vec{s}$$



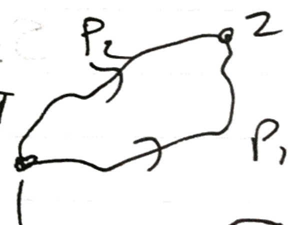
$$\int_1^2 \vec{F} \cdot d\vec{s} = \int_1^2 m\frac{d\vec{v}}{dt} \cdot \vec{v}\, dt$$

$$= \int_1^2 \frac{d}{dt}\left(\frac{1}{2} m \vec{v}^{\,2}\right) dt$$

$$= \frac{m}{2}\left(v_2^2 - v_1^2\right)$$

$$W_{12} = T_2 - T_1$$

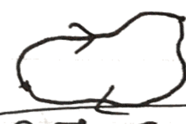
Consider $\vec{F} = \vec{F}(\vec{r})$



If W_{12} is path independent

$$W_{12}(P_1) = W_{12}(P_2)$$

$\Rightarrow$



$$\oint \vec{F} \cdot d\vec{s} = W_{12}(P_2) - W_{12}(P_1) = 0$$

$\boxed{\oint \vec{F} \cdot d\vec{s} = 0}$

Conservative force/system

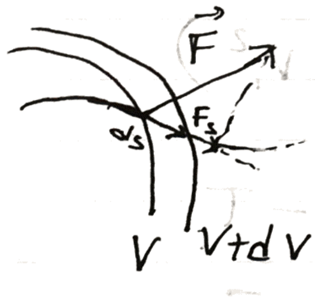
If $\vec{F} = \vec{F}(\vec{r})$ satisfied $\oint \vec{F} \cdot ds$

over any closed path

then $\vec{F}(\vec{r}) = -\nabla V(\vec{r})$ for some function $V(\vec{r})$.

$$\int_P \vec{F} \cdot ds = V(O) - V(r)$$

Defines $V(r)$ upto a constant.



$$\vec{F} \cdot d\vec{s} = F_s\, ds = -\frac{\partial V}{\partial s} ds$$

Since $W_{12} = T_2 - T_2$

and $W_{12} = V_1 - V_2$ for conservative systems

$$T_1 + V_1 = T_2 + V_2$$

<u>Conservation of Energy</u>

Forces conservative $\Rightarrow$ $T+V$ conserved -

System of Particles

Force on ith particle due to jth particle

$$\vec{F}_i = \sum_j \vec{F}_{ji} + \vec{F}_i^{(e)}$$

External force

Newton 3: $\vec{F}_{ij} = -\vec{F}_{ji}$

(Weak) Law of action & reaction

Note that then $\vec{F}_{ii} = 0$

$$\frac{d^2}{dt^2}\sum_i m_i \vec{r}_i = \sum_i \vec{F}_i^{(e)} + \underbrace{\sum_{ij} F_{ji}}_{\text{vanishes}}$$

Think of the two particle example $\sum_{ij} F_{ij} = F_{12} + F_{21}$

Define center of mass

$$\vec{R} = \frac{\sum m_i \vec{r}_i}{\sum_i m_i} = \frac{\sum m_i \vec{r}_i}{M}$$

M total mass

$$M \frac{d^2\vec{R}}{dt^2} = \sum_i \vec{F}_i^{(e)} = \vec{F}^{(e)}$$

[redacted] If we define total momentum to be

$$\vec{P} = M\frac{d\vec{R}}{dt} = \sum m_i \frac{d\vec{r}_i}{dt}$$

then

$$\frac{d\vec{P}}{dt} = \vec{F}^{(e)}$$

<u>Conservation theorem for total linear momentum</u>

[redacted] (Total) external force is zero [redacted] $\Rightarrow$ Total linear momentum is conserved

Interactions between the particles only lead to exchange of momentum among [redacted] themselves. The total momentum remains unaffected by those interactions.

What about total angular Momentum?

$$\frac{dL}{dt} = \frac{d}{dt}\sum_i \vec{r}_i \times \vec{p}_i = \sum_i \vec{r}_i \times \dot{\vec{p}}_i$$

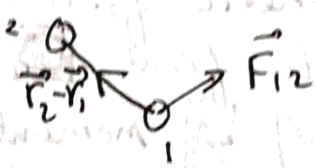
[redacted] $\frac{d}{dt} r_i \times p_i = v_i \times (m v_i) = 0$, remember? [redacted]

So

$$\frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \vec{F}_i = \sum_i r_i \times \vec{F}_i^{(e)}$$
$$+ \sum_i r_i \times \vec{F}_{ji}$$

How do we deal with this second term?
Look at the two particle case again.

$$\vec{r}_1 \times \vec{F}_{21} + \vec{r}_2 \times \vec{F}_{12} = (\vec{r}_2 - \vec{r}_1) \times \vec{F}_{12}$$



If $\vec{F}_{12}$ is in the direction of $\vec{r}_2 - \vec{r}_1$ (strong law of action and reaction) then $(\vec{r}_2 - \vec{r}_1) \times \vec{F}_{12} = 0$

In that case $\sum_{ij} \vec{r}_i \times \vec{F}_{ji}$

$$= \frac{1}{2} \sum_{ij} (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ji}$$

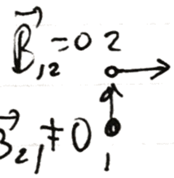
$$\Rightarrow \quad \frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \vec{F}_i^{(e)} = \vec{N}^{(e)} \leftarrow \text{External torque}$$

In that case, we have

<u>Conservation</u> theorem for total angular momentum! N total external torque

$\Rightarrow$ Angular momentum is conserved.

Before we push on, we should not that $\vec{F}_{ij} = -\vec{F}_{ji}$ and $\vec{F}_{ij} \times (\vec{r}_i - \vec{r}_i) = 0$ are not always true, e.g. magnetic forces / Biot-Savart law.

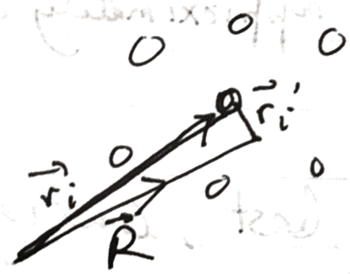


Physicist tend to find nontrivial generalizations of momentum/angular momentum/energy when things do not work out. For Biot-Savart law, one needs to invoke momentum/angular momentum of electromagnetic fields. The more sophisticated formulation of dynamics allows us to find such ~~conservation laws~~ generalization by relating conservation laws to symmetries.

Note that the total angular momentum could be broken up into the angular momentum of of all particles concentrated at the center of mass, plus the angular momentum of the particles around the center of mass.

$$\vec{r}_i = \vec{r}_i\,' + \vec{R} \qquad \vec{R} = \frac{\sum_i m_i \vec{r}_i}{\sum m_i}$$

$$\boxed{\sum_i m_i \vec{r}_i\,' = 0}$$



$$\vec{V}_i = \vec{v}_i\,' + \vec{V}$$

$$\vec{v}_i\,' = \frac{d\vec{r}_i\,'}{dt}, \quad \vec{V} = \frac{d\vec{R}}{dt}$$

$$\boxed{\sum_i m_i \vec{v}_i\,' = 0} \qquad \sum m_i \vec{v}_i\,' = p_i'$$

$$\vec{L} = \sum_i \vec{r}_i \times m_i \vec{v}_i = \sum_i \vec{R} \times m_i \vec{V}$$

$$+ \sum_i \vec{r}_i\,' \times m_i v_i'$$

$$+ \sum_i \vec{r}_i\,' \times (m_i \vec{V}) + \sum_i \vec{R} \times m_i \vec{v}_i\,'$$

Crossterms

$$\left(\sum_i m_i \vec{r}_i\,'\right) \times \vec{V} = 0 \qquad \vec{R} \times \sum m_i \vec{v}_i\,' = 0$$

$$\vec{L} = \vec{R} \times M\vec{V} + \sum_i \vec{r}_i\,' \times m_i \vec{v}_i\,'$$

$$\vec{L} = \vec{R} \times \vec{P} + \sum \vec{r}_i' \times \vec{p}_i'$$

'C.M' angular mom.

Angular mom. around C.M

Under some conditions they could each be approximately conserved.

Last but not the least, energy.

$$W_{12} = \sum_i \int_1^2 F_i \cdot ds_i = \overset{\sum \int m_i \dot{\vec{v}}_i \cdot \vec{v}_i dt}{\cancel{\ \ }} = \sum_i \int_1^2 d\left(\tfrac{1}{2} m_i v_i^2\right) = T_2 - T_1$$

Where $T = \frac{1}{2} \sum_i m_i v_i^2$

But $W_{12} = \sum_i \int_1^2 F_i \cdot ds_i$

$$= \sum_i \int_1^2 F_i^{(e)} \cdot ds_i + \sum_{ij} \int_1^2 \vec{F}_{ji} \cdot ds_i$$

If $\vec{F}_i^{(e)} = -\nabla_i V_i(\vec{r}_i)$

and $\vec{F}_{ji} = -\nabla_i V_{ij}(\vec{r}_i - \vec{r}_j)$,

with

$$V_{ij}(\vec{r}_i - \vec{r}_j) = V_{ji}(r_j - r_i)$$

[special case $V_{ij}(r_i - r_j) = V_{ij}(|r_i - r_j|)$]

then $\sum_i \int \vec{F}_i^{(e)} \cdot d\vec{s}_i = -\sum_i \int \nabla_i \vec{V}_i \cdot d\vec{s}_i$

$$-\frac{1}{2} \sum_{ij} \left(\nabla_i V_{ji} \cdot ds_i + \nabla_j V_{ij}\, ds_j \right)$$

The first term is just

$$-\sum_i V_i \Big|_1^2$$

The second one could be simplified using

$$\nabla_i V_{ji} \cdot ds_i + \nabla_j V_{ij} \cdot ds_j$$

$$\nabla_i V_{ji}(\bar{r}_i - \bar{r}_j) \cdot ds_i + \nabla_j V_{ji}(\bar{r}_j - \bar{r}_j) \cdot ds_j$$

$\nabla_i V_{ij} \cdot ds_i - \nabla_i V_{ij} \cdot ds_j = \nabla_i V_{ij} \cdot ds_i$

Then the second term is just

$$\frac{1}{2}\sum_{ij} V_{ij}(\vec{r}_i - \vec{r}_j)\Big|_1^2$$

So if $V = \sum_i V_i(\vec{r}_i) + \frac{1}{2}\sum_{ij} V_{ij}(\vec{r}_i - \vec{r}_j)$

Then once more $T + V\Big|_1^2 = 0$

or $T+V$ is conserved

$\to$ Theorem of Conservation of Energy for a system of particles.

Note that $T = \frac{1}{2}\sum m_i v_i^2$

$$= \frac{1}{2}\sum_i m_i (V + v_i')^2$$

$$= \frac{1}{2}MV^2 + \frac{1}{2}\sum m_i v_i'^2 + \sum_i m_i v_i' \cdot V$$

C.M. K.E. 'relative' K.E. $\sum_i m_i v_i' = 0$

So $T = \frac{1}{2}MV^2 + \frac{1}{2}\sum_i m_i v_i'^2$

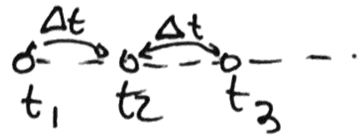
A brief word on numerically solving Newton's equation.

One dim: $\dot{x}(t) = v(t)$

$\dot{v}(t) = f(x(t))$ $\qquad f(x) = \frac{F(x)}{m}$

'Obvious' discretization

$$x(t_{n+1}) = x(t_n) + v(t_n)\,\Delta t$$
$$v(t_{n+1}) = v(t_n) + f(x(t_n))\,\Delta t$$



Problem; error $O(\Delta t)$, lack of reversibility, stability, ...

Cleverer discretization : Leapfrog method