Course: Classical Mechanics

Problem Set 2

Due: Nov 6, 2018 (5pm)

1. An alien spaceship of mass m is locked in an elliptic orbit of eccentricity e, around a planet with a mass much larger than the spaceship's. The planets mass distribution is spherically symmetric, making the potential $V(r) = -\frac{k}{r}$ away from the planet.

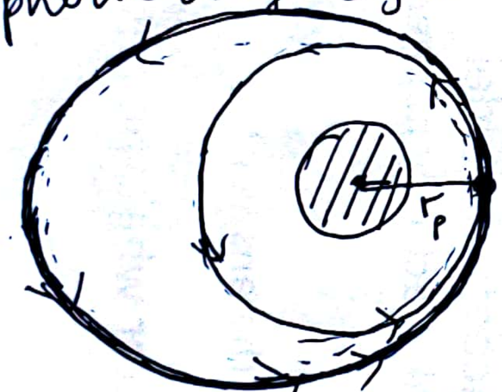


Fig 1. The two orbits

The captain wants to shift the orbit from elliptical to circular at the point of closest approach, $r = r_p$. She does it by thrusts creating an impulse altering the speed from v_i to v_f at $r = r_p$,

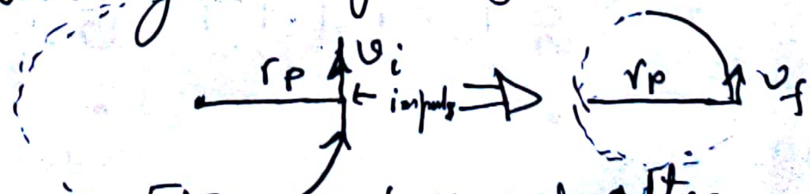
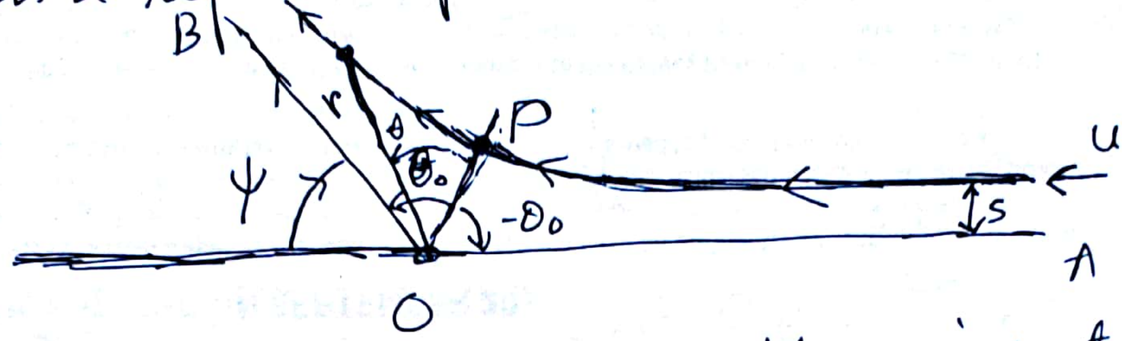


Fig2. Before and After

What is $\frac{v_f}{v_i}$?

2. Now consider a central force problem with a potential $V(r) = \frac{k}{r}$, with $k > 0$, meaning it is a repulsive potential. The origin O is at $r=0$.



The particle starts far away with an impact parameter s and speed u, moving parallel to $\overrightarrow{AO}$ and gets deflected to an asymptote parallel to $\overline{OB}$.

We set up a polar coordinate system, measuring θ from OP, P being the closest point on the trajectory to O. The hyperbolic trajectory is given by $\frac{1}{r} = \frac{mk}{l^2}(e\cos\theta - 1)$ where m is the reduced mass, l is the angular momentum and e is the eccentricity. The asymptotes are at angles θ_0 and $-\theta_0$ from OP.

a) Relate [redacted] energy E and angular momentum l to m, u, s.

b) Relate θ_0 to e, the eccentricity.

c) Relate the scattering angle [redacted] between $\vec{AO}$ and $\vec{OB}$, ψ, to θ_0.

d) [redacted] Express ψ in terms of [redacted] eccentricity e.

e) Express eccentricity e, in terms of E, l, m, k, which can then be written in terms of m, k, u and s.

f) [redacted] Plot, qualitatively, ψ as a function of s.

3. Consider Euler angles description of rigid rotation: $A(\psi,\theta,\phi) = B(\psi)\; C(\theta)\; D(\phi)$

$$= \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Now put θ to be π. $A(\psi,\pi,\phi)$ is a special (meaning determinant one) orthogonal matrix. Such a matrix would a rotation around some axis, according to Euler rotation theorem.

a) What is the axis of rotation?

b) What is the angle of rotation?

4. Let B be an antisymmetric real matrix. Let $e^B = \sum_{n=0}^{\infty} \frac{B^n}{n!}$. If $A = e^B$, show that A is an orthogonal matrix.

[Hint: start by showing that if B and C commute, $e^B e^C = e^{B+C}$]