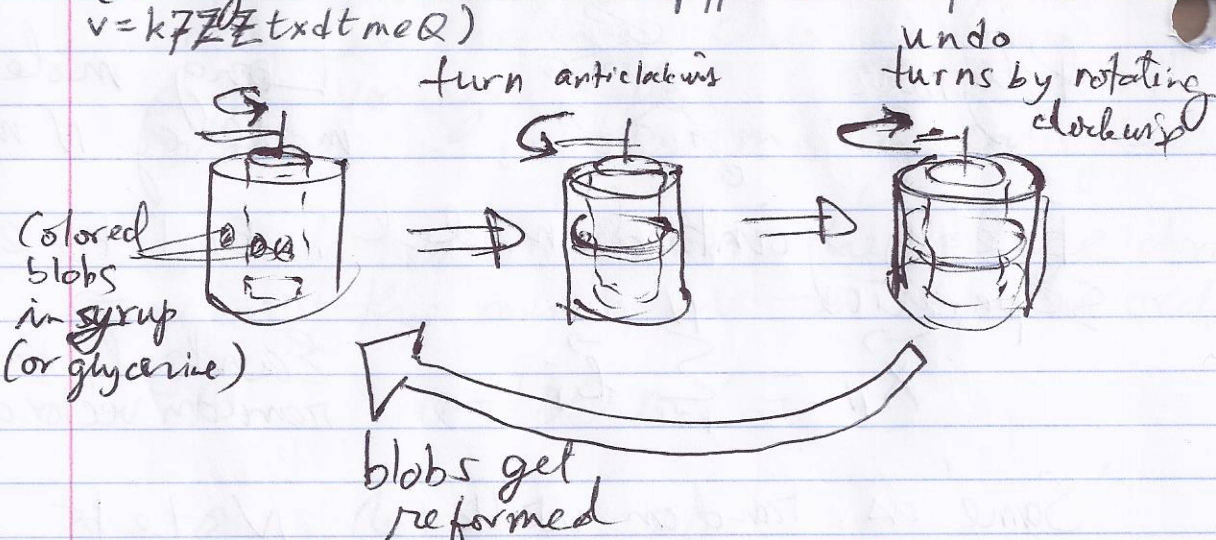
Low Reynolds Number Flow

We discussed drag and mentioned Stokes law $F_{drag} = 6\pi\eta a v$, η being the viscosity.

In this section we are going to look at the consequence of viscosity on motion at small scale.

An experiment on flow at 'high' viscosity (see youtube video at http://www.youtube.com/watch?v=k7ZZtxdtmeQ)

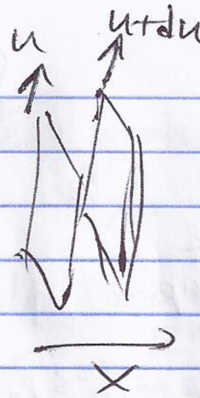


Nothing mixed!

(There would be a little bit of mixing because of difusion, but that is small in minutes time scale in a viscous liquid)

In water, this would not happen!

Viscosity

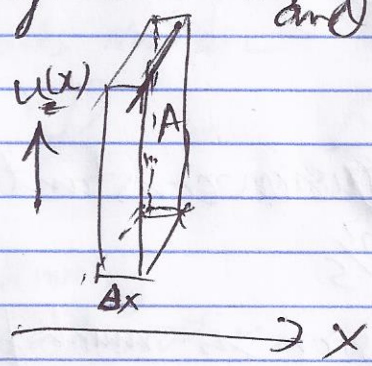


$$\frac{\text{Force}}{\text{Area}} = \eta \frac{du}{dx}$$

Newtonian Fluid

Viscosity and Momentum diffusion.

Imagine velocity to be only in z direction and it only depends on x



$$\rho A dx \frac{\partial u(x,t)}{\partial t} = -\eta A \frac{\partial u}{\partial x}(x,t) + \eta A \frac{\partial u}{\partial x}(x+dx,t) \simeq \eta A \Delta x \frac{\partial^2 u(x,t)}{\partial x^2}$$

$$\Rightarrow \quad \rho \frac{\partial u_z}{\partial t} = \eta \frac{\partial^2}{\partial x^2} u_z \Rightarrow \frac{\partial u_z}{\partial t} = \frac{\eta}{\rho} \frac{\partial^2 u_z}{\partial x^2}$$

Note the η/ρ plays the role of diffusion constant here. Since overall momentum is conserved (like mass or particle number) we get an equation that is similar to diffusion equation. We should later see the real equation, namely, the Navier-Stokes equation.

	$\rho\,(kg\,m^{-3})$	$\eta\,(Pa\,s)$
air	1	2×10^{-5}
water	10^3	8.9×10^{-4}
glycerine	1.3×10^3	1
Corn syrup	10^3	5

$$\left(\frac{\eta}{\rho}\right)_{water} \approx \frac{10^{-3}\,Pa\,s}{10^3\,kg/m^3} = 10^{-6}\,\frac{m^2}{s} = 10^{6}\,\frac{(\mu m)^2}{s}$$

$$\left(\frac{\eta}{\rho}\right)_{\text{corn syrup}} \approx \frac{5\,Pa\,s}{10^3\,kg/m^3} = 5\times10^{-3}\,\frac{m^2}{s} = 5\times10^{9}\,\frac{(\mu m)^2}{s}$$

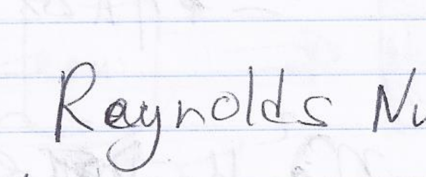
Small molecule diffusion constant in water $< 1\,\frac{cm^2}{s} = 10^4\,(\mu m)^2/s$

Momentum diffusion is much faster in these fluids than material diffusion.

Reynolds Number:

Viscosity smooths out velocity gradients. Inertia likes to keep velocities to be the same.

When does viscosity win over inertia?



If there is a velocity profile changing over distance R, with velocity v:

Particles would [redacted] move through distance R in time $\sim R/v$

Viscosity [redacted] could smooth out the velocity gradiant in time $\sim \dfrac{R^2}{\eta/\rho}$

[redacted] Viscosity [redacted] wins is the second time is fraction of the first time scale.

Reynolds Number

$$R_e = \frac{R^2/(\eta/\rho)}{R/v} = \frac{\rho R v}{\eta}$$

Swimming in [redacted] water $R_e = \dfrac{\rho R v}{\eta} = \dfrac{R v}{\eta/\rho} = \dfrac{R v}{10^{-6}\ m^2/s}$

[redacted] Humans $R \sim 1m$ $v \sim 1 m/s$

$$Re = \frac{1m\ 1m/s}{10^{-6}\ m^2/s} = 10^6$$

← Flow dominated by eddies, turbulence

E. Coli $R = 1\mu m$ $v \sim 10\ \mu m/s$

$$Re = \frac{10^{-6}m\ 10^{-5}m/s}{10^{-6}m^2/s} = 10^{-5}$$

↑ Laminar flow

Navier-Stokes Equation for Incompressible Fluids

$$\rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\vec{\nabla}\vec{u}\right)$$

$$= -\vec{\nabla} p + \eta \nabla^2 \vec{u} + \vec{f}$$

along with $\vec{\nabla}\cdot\vec{u} = 0$

Different parts

$$\underbrace{\rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\vec{\nabla}\vec{u}\right)}_{\text{mass x acceleration}} = \underbrace{-\vec{\nabla} p}_{\text{force from pressure gradient}} + \eta \nabla^2 \vec{u} + \vec{f}$$

~~force~~ force from pressure gradient

non uniformity of viscous force

Velocity change in the Lagrangian fram : Convective acceleration

body force density