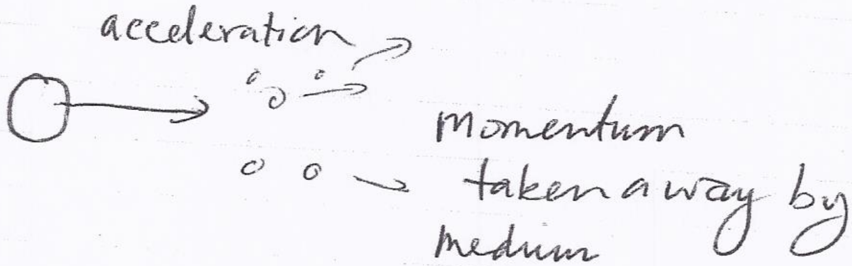
Diffusion and Dissipation

What is [redacted] the [redacted] relation between the "force" causing the bias and the Velocity?



$\Rightarrow$ terminal velocity

$$v_{drift} = \overset{\text{mobility}}{\mu} F \quad \text{or} \quad F_{drad} = -\overset{\text{drag coeff}}{\zeta} v$$

$F + F_{drag} = 0$ in the long run.

[redacted] On the average

$$m\frac{dv(t)}{dt} = F(t) - \zeta v$$

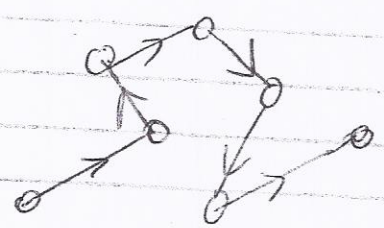
$$v(t) = v(0)\, e^{-\frac{\zeta t}{m}} + \int_0^t dt'\, e^{-\frac{\zeta(t-t')}{m}} \frac{F(t')}{m}\, dt'$$

when $t \gg \frac{m}{\zeta} = \tau$ and [redacted] F is a constant

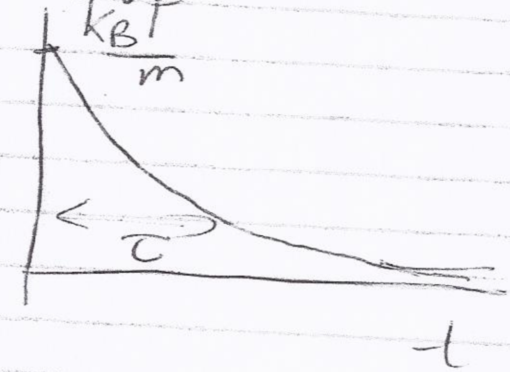
$$v = \frac{F}{m}\int_0^{\infty} e^{-\frac{\zeta s}{m}}\, ds = \frac{F}{\zeta} \qquad \boxed{\zeta = \mu^{-1}}$$

The time scale τ is associated with velocity correlation decay.

$C(t) = \langle v_x(t)\, v_x(0) \rangle$

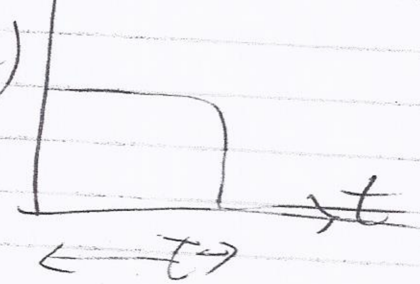


$C(t)$



We will pretend that τ is collision time and velocity is completely equilibriated at the collision. As if

$C(t)$



Turns out that the conclusions are valid in general, as we will show.

Let us say that just after a collision the x component of the velocity is $v_{0,x}$
With force being applied for time Δt

$$\Delta x = v_{0,x}\Delta t + \frac{F}{2m}(\Delta t)^2$$

The 'extra' velocity is $\frac{F}{2m}\frac{(\Delta t)^2}{\Delta t} = \frac{F}{2m}\Delta t$

If τ is the time scale, after which velocities get 'reset', then

$$v_{drift} = \frac{F\tau}{2m} \Rightarrow \boxed{\frac{1}{\zeta} = \mu = \frac{\tau}{2m}}$$

What about diffusion?

Set $F=0$

$$\Delta x = v_{0,x}\,\Delta t$$

In time t there are $\frac{t}{\tau}$ periods of motion in one direction

$$\langle x^2 \rangle = \sum \langle \Delta x^2 \rangle = \frac{t}{\tau}\langle v_{0,x}^2 \rangle \tau^2$$

$$= \langle v_{0,x}^2 \rangle \tau t \equiv 2Dt$$

So $$\boxed{D = \frac{\langle v_{0,x}^2 \rangle \tau}{2}}$$

Note both D and μ involve τ, something to do with kinetics not just thermal equilibrium distribution

Note that

$$\zeta D = \frac{D}{\mu} = m\langle v_{0,x}^2 \rangle$$

has lost the timescale of kinetics. At thermal equilibrium

$$m\langle v_{0,x}^2 \rangle = k_B T$$

according to Equipartition Theorem.

Hence

$$\boxed{\zeta D = \frac{D}{\mu} = k_B T}$$

Einstein Relation

or

$$\boxed{D = \mu k_B T}$$

What is the drag coefficient? For 'mesoscopic/macroscopic' objects, we could use Stokes law for spherical objects

$$\boxed{F = 6\pi\eta a v}$$

η is the viscosity of the medium and a is the radius.

$$\eta = 8.9 \times 10^{-4} \text{ Pa s} \quad \text{around room temp.}$$

For a $1 \mu m$ radius bead.

$$\zeta = 6\pi\eta a = 6 \times 3.14 \times 8.9 \times 10^{-4} \times 10^{-6} \frac{N}{m/s}$$
$$= 1.7 \times 10^{-8} \frac{N}{m/s}$$

$$D = \frac{k_B T}{\zeta} = \frac{4.1 \times 10^{-21} \text{ Nm}}{1.7 \times 10^{-8} \text{ Ns/m}} = 2.4 \times 10^{-13} \frac{m^2}{s}$$
$$= 0.24 \frac{(\mu m)^2}{s}$$

This is what makes Brownian motion observable. Over ten's of seconds an pollen sized object moves over microns.

If we go down to protein molecule sized objects $a \sim 1$-$10 nm$, D should be 100-1000 times bigger! In practice, inside the cell, the observed diffusion constant is orders of magnitude smaller, because cytosol is not just water!