Probability Distributions Redux.

Binomial Distribution

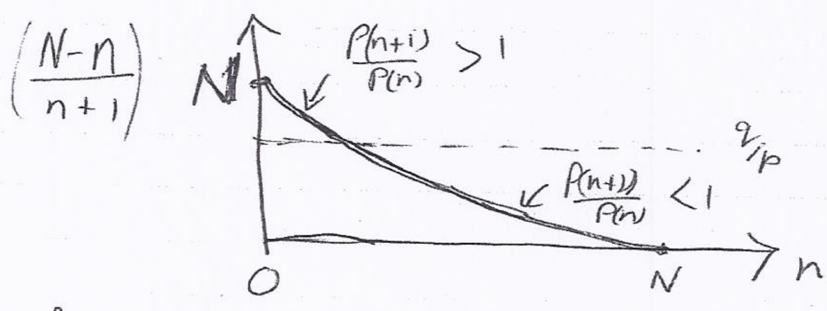
$$P(n) = \binom{N}{n} p^n q^{N-n}$$

N trials, p=probability of success, $q=1-p$

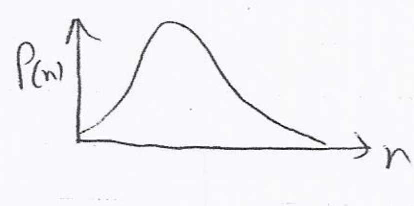
n successes.

$$\frac{P(n+1)}{P(n)} = \frac{\frac{N!}{(n+1)!(N-n-1)!} \quad p^{n+1} q^{N-n-1}}{\frac{N!}{n!(N-n)!} \quad p^n q^{N-n}}$$

$$= \frac{N-n}{n+1} \frac{p}{q}$$



When $q/p < N$



$$(N+1)p - 1 = Np - q \quad \text{is the threshold.}$$

The integer above is the max, unless $Np-q=\tilde{n}$ is an int. then $P(\tilde{n})$ and $P(\tilde{n}+1)$ are equal.

Moments and Generating Function

$$G(z) = \langle z^n \rangle = \sum_n z^n P(n)$$

Note that $G(1) = \sum_n P(n) = 1$

Now the [redacted] fun begins!

$$z\frac{d}{dz} G(z) = \sum_n z\frac{d}{dz}(z^n) P(n) = \sum_n n z^n P(n)$$

$$z\frac{d}{dz} G(z)\Big|_{z=1} = \sum_n n P(n) = \langle n \rangle$$

Similarly

$$\left(z\frac{d}{dz}\right)^K G(z)\Big|_{z=1} = \langle n^k \rangle$$

Let us put it to practice.

Binomial Distr.^{n}

$$G(z) = \sum_{n=0}^{N} z^n \binom{N}{n} p^n q^{N-n}$$

$$= \sum_{n=0}^{N} \binom{N}{n} (zp)^n q^{N-n}$$

$$= (zp+q)^N$$

Check $G(1) = (p+q)^N = 1$

$$z \frac{d}{dz} G(z) = z \frac{d}{dz} (zp+q)^N = Nzp(zp+q)^{N-1}$$

At $z=1$, we get Np. So $\langle n \rangle = Np$.

$$\left(z\frac{d}{dz}\right)^2 G(z) = z\frac{d}{dz}\left[Nzp(zp+q)^{N-1}\right]$$

$$= Np(zp+q)^{N-1} + (N-1)Nzp^2(zp+q)^{N-2}$$

Setting $z=1$

$$\langle n^2 \rangle = Np(p+q)^{N-1} + (N-1)Np^2(p+q)^{N-2}$$

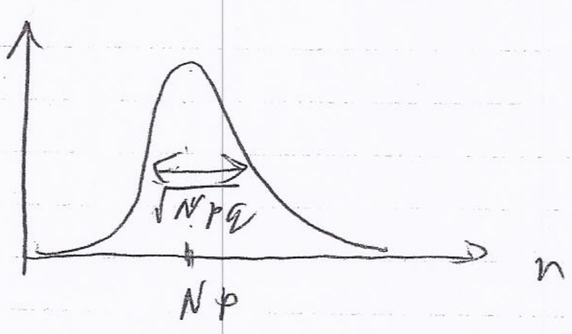
$$= Np + N^2p^2 - Np^2 = Np(1-p) + N^2p^2$$

$$= (Np)^2 + Npq$$

$$\mathrm{Var}(n) = \langle (n - \langle n \rangle)^2 \rangle = \langle n^2 \rangle - 2\langle n \rangle\langle n \rangle + \langle n \rangle^2$$

$$= \langle n^2 \rangle - \langle n \rangle^2 = (Np)^2 + Npq - (Np)^2$$

$$= Npq$$



When N is large
st. dev << mean

$$\frac{\text{st. dev}}{\text{mean}} = \frac{\sqrt{\mathrm{Var}}}{\text{mean}} = \frac{\sqrt{Npq}}{Np} = \sqrt{\frac{q}{Np}} \sim \frac{1}{\sqrt{N}}$$

Poisson Distribution

When $N \to \infty$ and $p \to 0$ with $Np = \nu$ fixed

$$P(n) = \binom{N}{n} p^n q^{N-n} = \frac{N!}{n!(N-n)!} p^n q^{N-n}$$

$$= \frac{N(N-1)\cdots(N-n+1)}{n!} \left(\frac{p}{1-p}\right)^n (1-p)^N$$

$$= 1\left(1-\tfrac{1}{N}\right)\cdots\left(1-\tfrac{n-1}{N}\right) \times \frac{1}{n!} \cdot \left(\frac{Np}{1-p}\right)^n \cdot e^{N\ln(1-p)}$$

As $N \to \infty$, $p \to 0$ with $Np = \nu$

$$P(n) = 1\left(1-\tfrac{1}{N}\right)\left(1-\tfrac{2}{N}\right)\cdots\left(1-\tfrac{n-1}{N}\right)\frac{1}{n!}\left(\frac{\nu}{1-\frac{\nu}{N}}\right)^n e^{N\ln\left(1-\frac{\nu}{N}\right)}$$

$$= \left(1+o\left(\tfrac{1}{N}\right)\right) \frac{\nu^n}{n!} e^{-\nu\left(1+o\left(\frac{1}{N}\right)\right)}$$

$$\to \frac{\nu^n}{n!} e^{-\nu}$$

$$\frac{P(n+1)}{P(n)} = \frac{\nu}{n+1}$$

$P(n)$ peak at $[\nu]$ (the floor of ν)

$$\mathrm{Var}(n) = \lim_{\substack{N \to \infty \\ p \to 0 \\ \text{with } Np \text{ fixed}}} Npq = \lim_{N \to \infty} N \frac{\nu}{N}\left(1 - \frac{\nu}{N}\right)$$

$$= \nu$$

So the mean is ν and the fluctuation around is $\sqrt{\nu}$.

$$G(z) = \sum_{n=0}^{\infty} z^n P(n)$$

Properties of Generating functions when independent variables are added.

x distr$\underline{n}$ $\quad P_1(x) \to \langle z^x \rangle = G_1(x)$

y $\quad$ " $\quad P_2(y) \to \langle z^y \rangle = G_2(y)$

$$\mathrm{Prob}(x, y) = P_1(x)\, P_2(y)$$

$$w = x + y$$

$$G(z) = \langle z^w \rangle = \sum_{x, y} z^{x+y} P_1(x) P_2(y)$$

$$= \sum_x z^x P_1(x) \sum_y z^y P_2(y) = G_1(z)\, G_2(z)$$