Stokes Flow

For many problems at small length scales, simplifications are possible.

First, compare $\rho \vec{u}\cdot\nabla u$ to $\eta \nabla^2 u$.

Note

$$\rho u \cdot \nabla u \sim \rho \frac{u^2}{R} \qquad \eta \nabla^2 u \sim \eta u/R^2$$

If $\eta u/R^2 \gg \rho u^2/R$, viscosity dominates.
~~This is~~ condition $1 \gg \frac{\rho u R}{\eta} = Re$, namely the low Reynolds number limit.

$$\rho \frac{\partial \vec{u}}{\partial t} = -\vec{\nabla} p + \eta \nabla^2 \vec{u} + \vec{f}$$

(Linearized Navier-Stokes)

For many problems in biology ~~[illegible]~~ at the cellular level, forcing changes over time scales that ~~which~~ are longer than the time to ~~diffuse~~ have ~~the~~ forces in equilibrium around the object. In other words, if forcing happens at timescale τ, and the system size is R, we often have

$$\tau \gg R^2/(\eta/\rho) \leftarrow \text{time for momentum 'to diffuse' through the system.}$$

In that case we can drop the $\rho \frac{\partial \vec{u}}{\partial t}$ term.

Quasi-steady-state approximation

$$-\vec{\nabla} p + \eta \nabla^2 \vec{u} + \vec{f} = 0$$

Stokes equation!

This equation is supplemented by the incompressibility condition

$$\vec{\nabla} \cdot \vec{u} = 0$$

Four equations for four variables $p, \vec{u}$.

Biological Applications

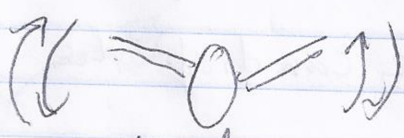
Swimming:

$f(t) \rightarrow u(t)$ linear mapping.

$f \rightarrow -f \qquad u \rightarrow -u$

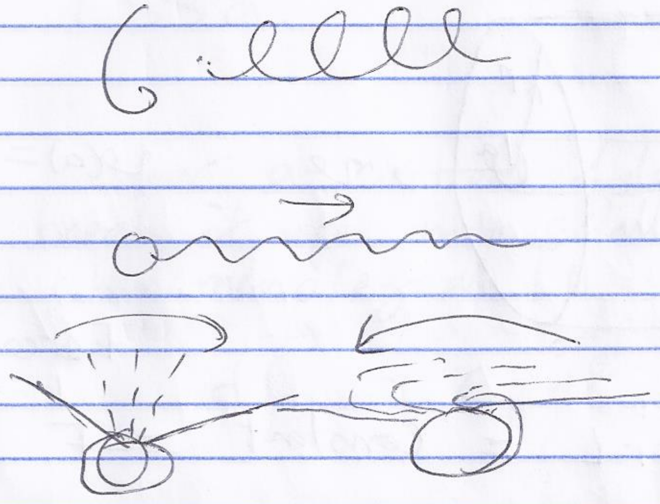
The rate does not matter.

If motion is reciprocal.



Net motion is cancelled

Need something "non-reciprocal":

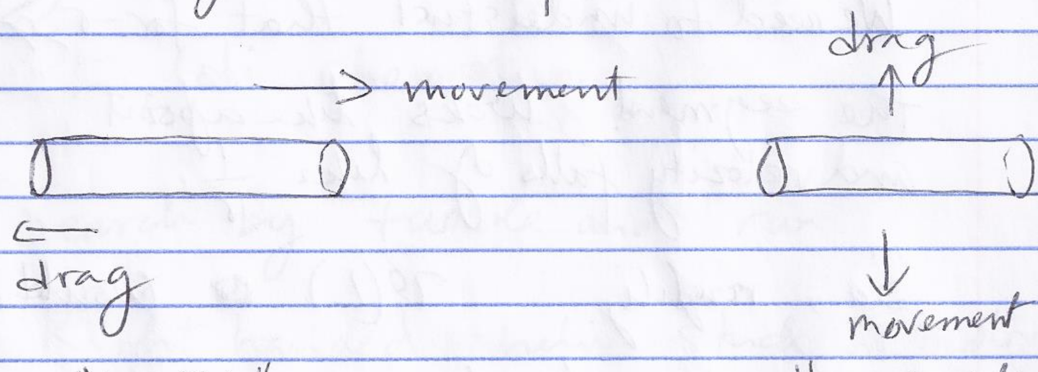


Flagella rotating

Waves on sperm tails

Whipping motion of cilia

Drag on thin filament.

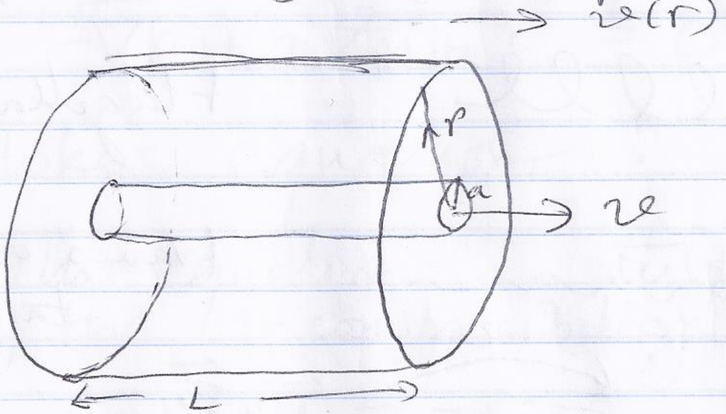


"parallel" "perpendicular"

$\perp$ drag $>$ $\parallel$ drag

The drag 'coefficients' have non-trivial dependence of the size and shape.

~~Quasi steady-state approximation~~



$v(a) = v$

Force on rod

$$\eta \frac{dv(r)}{dr}\, 2\pi r L = \text{constant} = -F$$

Problem! $\quad v(r) \sim -\frac{F}{2\pi\eta L}\log\frac{r}{a} + v$

We need to understand that for $r \gg L$ the segment looks like a point and velocity falls of like $\frac{1}{r}$.

So, roughly $\quad v(L)$ would be small

$$v(L) = v - \frac{F}{2\pi\eta L}\log\frac{L}{a} \simeq 0$$

$$\Rightarrow F = \frac{2\pi\eta L v}{\log\left(\frac{L}{a}\right)}$$

So $\quad F_{\parallel} = \frac{2\pi\eta L v}{\log\frac{L}{a}}$

Turns out $F_\perp = \frac{4\pi \eta L v}{\log\left(\frac{L}{a}\right)}$ when $L >> a$.

For a curved filament, it is common to break it up into smaller segments and use phenomenological $\parallel$ and $\perp$ drag coefficients

$$d\vec{f}_\parallel = c_\parallel \vec{v}_\parallel \, ds$$

length of the arc

$$d\vec{f}_\perp = c_\perp \vec{v}_\perp \, ds$$

E. coli chemotaxis

Search by tumble and run

Right handed helix stuck to motors

Counterclockwise rotn
(CCW)