Mathematical Physics 2019
Midterm Solutions

1 c) Let $z = x+iy$ $\quad \tau = \sigma + i\mu$

$a_n = e^{i\pi n^2\tau + 2\pi i n z}$ $\quad x, y, \sigma, \mu \in \mathbb{R},\ \mu > 0$

$$|a_n| = \left| e^{i\pi n^2\tau + 2\pi i n z} \right| = \left| e^{i\pi n^2\sigma + 2\pi i n x} \right| e^{-\pi n^2\mu - 2\pi n y} \text{ [redacted]}$$

$$= e^{-\pi n^2 \mu - 2\pi n y}$$

Let us organize $n_k = 0, +1, -1, +2, -2, \ldots$ for $k = 1, 2, 3, \ldots$

So $\frac{k}{2} - 1 \leq |n_k| \leq k/2$

$$|a_{n_k}|^{\frac{1}{k}}$$

$$= e^{-\pi \frac{n_k^2}{k}\mu - 2\pi \frac{n_k}{k} y}$$

$\left|\frac{n_k y}{k}\right|$ [redacted] $\leq \frac{1}{2}|y|$

[redacted] $\pi \frac{n_k^2}{k}\mu + 2\pi \frac{n_k}{k} y \geq \pi \frac{(k-\frac{1}{2})^2}{k}\mu -$ [redacted] $\pi\ |y|$

As $k \to \infty$ RHS [redacted] goes to $+\infty$

So $|a_{n_k}|^{\frac{1}{k}} \to 0$

making the series absolutely convergent.

It also means any rearrangement of the series is also convergent.

b) $\displaystyle\sum_{n=-\infty}^{\infty} e^{i\pi n^2\tau + 2\pi i n(z+1)}$ [redacted]

$= \displaystyle\sum_{n=-\infty}^{\infty} e^{i\pi n^2\tau + 2\pi i n z}$ $\quad$ for $f(z+1, \tau) = f(z, \tau)$

c) Putting $n \to n+1$ in the $\overbrace{\text{expression}}^{\text{summand}}$ just rearranges the series; which should converge to the same limit

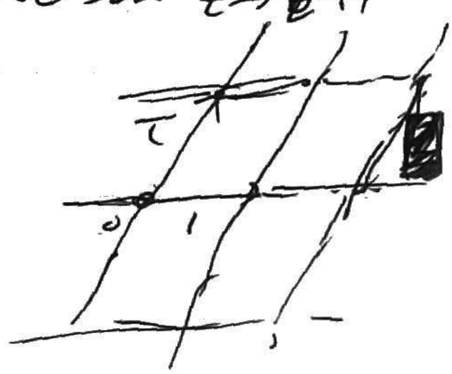
$$\sum_{n=-\infty}^{\infty} e^{i\pi n^2\tau + 2\pi i n z} = \sum_{n=-\infty}^{\infty} e^{i\pi(n+1)^2\tau + 2\pi i(n+1)z} = \sum_{n=-\infty}^{\infty} e^{i\pi(n^2+2n+1)\tau + 2\pi i(n+1)z}$$

$$= e^{i\pi\tau + 2\pi i z}\sum_{n=-\infty}^{\infty} e^{i\pi n^2\tau + 2\pi i n(z+\tau)}$$

$$\Rightarrow \quad f(z,\tau) = e^{i\pi\tau + 2\pi i z} f(z+\tau,\tau)$$

$$f(z+\tau,\tau) = e^{-i\pi\tau - 2\pi i z} f(z,\tau)$$

This is a theta function, periodic in $z \to z+1$ but quasiperiodic in $z \to z+\tau$



2. a) If rank of A is m, [redacted] then $im(A)$ is an m dimensional subspace of W. Consider $im(A)^{\perp}$, which [redacted] has dimension $n-m$.

$y \in im(A)^{\perp}$ iff $\langle y, Ax \rangle = 0$ for all [redacted] $x \in V$

iff $\langle A^{+}y, x \rangle = 0$, [redacted] for all $x \in V$

iff $A^{+}y = 0$

So $\ker(A^{+}) = im(A)^{\perp}$

nullity $= \dim \ker(A^{+}) = n-m$

b) $A^{+}A = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}$ $\qquad AA^{+} = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 4 \end{pmatrix}$

$A^{+}A$ eigenvectors are $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

with eigenvalues 6, 4.

AA^{+} eigenvectors are $\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

with eigenvalues 6, 4, 0

c) If $A^{+}Av = \lambda v$ for some $v \in V$, $v \neq 0$ and $\lambda \neq 0$ (we assumed it is so for all eigenvalues)

then Av is a eigenvector of A^{+}

since $A^{+}(Av) = \lambda (A^{+}v)$. Of course $Av \neq 0$

since, otherwise $\lambda v = 0$. Note the [redacted] corresponding eigenvalue is the same

For each λ_i we have $\phi_i \in V$ with

$\langle \phi_i , \phi_j \rangle_V = \delta_{ij}$.

$\langle A\phi_i , A\phi_j \rangle = \langle \phi_i , A^{+}A\phi_j \rangle = \lambda_j \langle \phi_i , \phi_j \rangle = \lambda_j \delta_{ij}$

So there are m independent. orthonormal eigenvectors of AA^{+}, given by $\frac{1}{\sqrt{\lambda_i}} A\phi_i$ with eigenvalue λ_i.

We also have $im(A)^{\perp}$ with an $n-m$ size orthonormal basis each of which is annihilated by AA^{+}. So those are eigenvectors with eigenvalues 0.

So we have $\lambda_1, \ldots, \lambda_m$ and $n-m$ zero eigenvalues

d) Use orthogonality of $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ and $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

and of $\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$ and $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

This is the singular value decomposition of A.

3. I apologize here! Symmetrizn needed.

$$g = \tfrac{1}{2} g_{\mu\nu}\, dx^\mu \otimes dx^\nu = \frac{\tfrac{1}{2}(dx \otimes dy + dy \otimes dx)}{1 + xy}$$

but you had all the components of the symmetric tensor.

a) $$g'_{\rho\sigma} = g_{\mu\nu} \frac{\partial x^\mu}{\partial x'^\rho} \frac{\partial x^\nu}{\partial x'^\sigma}$$

$$= g_{xy} \frac{\partial x}{\partial x'^\rho} \frac{\partial y}{\partial x'^\sigma} + g_{yx} \frac{\partial y}{\partial x'^\rho} \frac{\partial x}{\partial x'^\sigma}$$

$$= \frac{1}{1+xy}\left(\frac{\partial x}{\partial x'^\rho} \frac{\partial y}{\partial x'^\sigma} + (\rho \leftrightarrow \sigma)\right)$$

$x = e^{r+t} \qquad y = e^{r-t}$

$\dfrac{\partial x}{\partial r} = e^{r+t} \qquad \dfrac{\partial x}{\partial t} = e^{r+t}$

$\dfrac{\partial y}{\partial t} = e^{r+t} \qquad \dfrac{\partial y}{\partial t} = -e^{r-t}$

$g_{xy} = \dfrac{1}{1+xy} = \dfrac{1}{1+e^{2r}}$

$g_{rr} = 2\, g_{xy} \dfrac{\partial x}{\partial r}\dfrac{\partial y}{\partial r} = \dfrac{2e^{2r}}{1+e^{2r}}$

$g_{tt} = 2\, g_{xy} \dfrac{\partial x}{\partial t}\dfrac{\partial y}{\partial t} = -\dfrac{2e^{2r}}{1+e^{2r}}$

$g_{rt} = g_{xy}\left(\dfrac{\partial x}{\partial p}\dfrac{\partial y}{\partial t} + \dfrac{\partial x}{\partial t}\dfrac{dy}{dr}\right) = g_{xy}\left(-e^{2r} + e^{2r}\right) = 0$

$g_{tr} = 0$ by symmetry

So $g = \dfrac{e^{2r}}{1+e^{2r}}\left(dr \otimes dr - dt \otimes dt\right)$

b) $v = \frac{1}{2}\left(x\dfrac{\partial}{\partial x} - y\dfrac{\partial}{\partial y}\right) = \frac{1}{2}x\left(\dfrac{\partial r}{\partial x}\dfrac{\partial}{\partial r} + \dfrac{\partial t}{\partial x}\dfrac{\partial}{\partial t}\right)$
$\qquad - \frac{1}{2}y\left(\dfrac{\partial r}{\partial y}\dfrac{\partial}{\partial r} + \dfrac{\partial t}{\partial y}\dfrac{\partial}{\partial t}\right)$

Note $r = \frac{1}{2}\ln xy \qquad t = \frac{1}{2}\ln\dfrac{x}{y}$

$\dfrac{\partial r}{\partial x} = \dfrac{1}{2x} = \frac{1}{2}e^{-(r+t)}, \; \dfrac{\partial r}{\partial y} = \frac{1}{2}e^{-(r-t)}, \; \dfrac{\partial t}{\partial x} = \frac{1}{2}e^{-(r+t)}, \; \dfrac{\partial t}{\partial y} = -\frac{1}{2}e^{-(r-t)}$

$$v = \frac{1}{2} e^{r+t} \left[\frac{1}{2} e^{-r-t} \frac{\partial}{\partial r} + \frac{1}{2} e^{-r-t} \frac{\partial}{\partial t} \right]$$

$$- \frac{1}{2} e^{r-t} \left[\frac{1}{2} e^{-r+t} \frac{\partial}{\partial r} \text{[redacted]} - \frac{1}{2} e^{-r+t} \frac{\partial}{\partial t} \right]$$

$$= \frac{1}{2} \frac{\partial}{\partial t}$$

c) [redacted] $(\mathcal{L}_v g)_{\mu\nu} = v^\rho \dfrac{\partial g_{\mu\nu}}{\partial x^\rho} + g_{\mu\rho} \dfrac{\partial v^\rho}{\partial x^\nu} + g_{\rho\nu} \dfrac{\partial v^\rho}{\partial x^\mu}$

$v^\rho \dfrac{\partial g_{\mu\nu}}{\partial x^\rho}$ contributes only for $\mu, \nu = x, y$ or y, x

and it is symmetric in μ, ν.

[redacted]

$$v^\rho \frac{\partial g_{xy}}{\partial x^\rho} = \frac{1}{2}\left(x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right) \frac{1}{1+xy}$$

$$= \frac{1}{2} x \left(- \frac{y}{(1+xy)^2} \right) - \frac{1}{2} y \left(- \frac{x}{(1+xy)^2} \right)$$

$$= 0$$

$g_{\mu\rho} \dfrac{\partial v^\rho}{\partial x^\nu}$ contributes only for [redacted] $g_{xy} \dfrac{\partial v^y}{\partial y}$

$$= \frac{1}{1+xy}\left(-\frac{1}{2} \right)$$

$\hookrightarrow$ contributes to $(\mathcal{L}_v g)_{xy}$

and for $g_{yx} \dfrac{\partial v^x}{\partial x} = \dfrac{1}{1+xy}\left(\dfrac{1}{2}\right)$ $\rightarrow$ contributes to $(\mathcal{L}_v g)_{yx}$

$g_{\rho\nu} \dfrac{\partial v^\rho}{\partial x^\mu}$ contribute for

$(\mathcal{L}_v g)_{yx} \xleftarrow{\text{contr.}} g_{yx} \dfrac{\partial v^y}{\partial y} = \dfrac{1}{1+xy}\left(-\dfrac{1}{2}\right)$

$(\mathcal{L}_v g)_{xy} \xleftarrow{\text{contr}} g_{xy} \dfrac{\partial v^x}{\partial x} = \dfrac{1}{1+xy}\left(\dfrac{1}{2}\right)$

$(\mathcal{L}_v g)_{xx} = 0 \qquad (\mathcal{L}_v g)_{xy} = 0 + \dfrac{1}{1+xy}\left(-\dfrac{1}{2}\right) + \dfrac{1}{1+xy}\left(\dfrac{1}{2}\right)$

$(\mathcal{L}_v g)_{yy} = 0 \qquad = 0$

Similarly $(\mathcal{L}_v g)_{yx} = 0$

$v^r = 0 \quad v^t = \dfrac{1}{2}$. so no $\dfrac{\partial v^\rho}{\partial x^\mu}$.

$v^t \dfrac{\partial}{\partial t} g'_{\rho\sigma} = 0$ since the components do not depend on t. in the r,t coordinate. $\left(g_{rr} = \dfrac{e^{2r}}{1+e^{2r}},\ g_{tt} = -\dfrac{e^{2r}}{1+e^{2r}} \right)$

$g_{rt} = 2 \cdot 0 = 0$

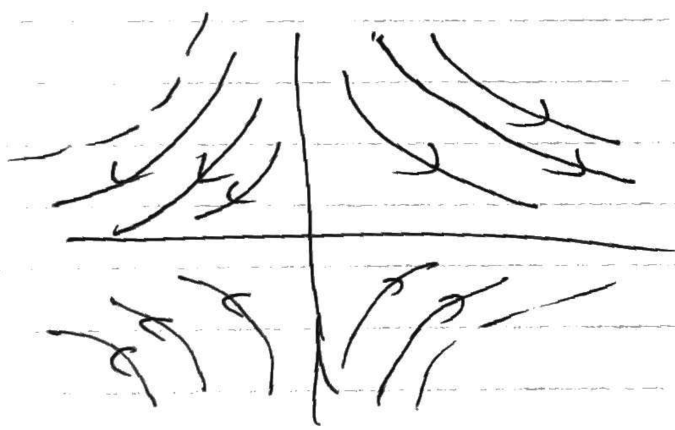
So $\mathcal{L}_v g = 0$ there too, as expected.

d) $\dfrac{dx}{d\lambda} = \frac{1}{2}x \qquad \dfrac{dy}{d\lambda} = -\frac{1}{2}y$

$x(\lambda) = e^{\lambda [redacted]/2}x(0) \qquad y(\lambda) = e^{-\lambda/2}y(0)$

are the integral curves.

Notice that $x(\lambda)\,y(\lambda) = x(0)\,y(0)$



In fact $x \to e^{\frac{\lambda}{2}}x$, $y \to e^{-\frac{\lambda}{2}}y$

is a symmetry of the system.

e) $\mathcal{L}_v \Omega$ [redacted] is an antisymmetric tensor with only nontrivial components bein $(\mathcal{L}_v\Omega)_{xy} = -(\mathcal{L}_v\Omega)_{yx}$

$$(\mathcal{L}_v\Omega)_{xy} = \frac{1}{2}\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)\Omega_{xy} + \Omega_{x\rho}\frac{\partial v^\rho}{\partial y} + \Omega_{\rho y}\frac{\partial v^\rho}{\partial x}$$

$$= \frac{1}{2}\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)\frac{1}{1+xy}$$

$$+ \Omega_{xy}\frac{\partial v^y}{\partial y} + \Omega_{xy}\frac{\partial v^x}{\partial x}$$

$$= 0 + \Omega_{xy}\left(-\tfrac{1}{2}\right) + \Omega_{xy}\left(\tfrac{1}{2}\right)$$

$$= 0$$

The Lie derivatives vanishing has to do with symmetry of the metric and the associated area/volume form. v is a so-called Killing vector field, generator of 'isometries' for the pseudo-Riemannian metric of signature $(1, 1)$, which happens to be a toy black-hole solution found in non-critical string theory.