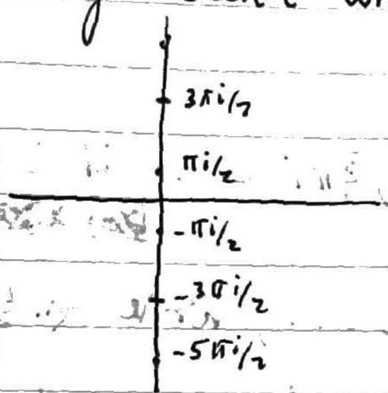
1 a) $\cosh z = 0 \Rightarrow e^{z} + e^{-z} = 0$

$$\Rightarrow e^{2z} = -1 = e^{(2n+1)\pi i} \qquad n \in \mathbb{Z}$$

So $z = (n+\frac{1}{2})\pi i = z_n$, $n \in \mathbb{Z}$, are where the poles of $\operatorname{sech} z$ are.



Near $z = z_n$, $\cosh z = \cosh z_n + \sinh z_n (z - z_n) + O((z-z_n)^2)$

$$= (\sinh z_n)(z - z_n) + O((z-z_n)^2)$$

$$\sinh z_n = \frac{e^{(n+\frac{1}{2})\pi i} - e^{-(n+\frac{1}{2})\pi i}}{2}$$

$$= \frac{(-1)^n i - (-1)^{-n}(-i)}{2}$$

$$= (-1)^n i$$

So $\operatorname{sech} z = \frac{1}{(-1)^n i (z - z_n)}$ + regular part near $z - z_n$

$$= \frac{i(-1)^{n+1}}{z - z_n} + \text{regular part}$$

Thus, at z_n, we have a simple pole, with residue $i(-1)^{n+1}$.

b) The function g has singularities whenever $z^2 + (n+\frac{1}{2})^2 \pi^2 = 0$ So $z = \pm(n+\frac{1}{2})\pi i$ $n = 0, 1, \cdots$ are the singularities. It is the same set of points.

Let us look at the ~~residues~~

$$\frac{(-1)^n(2n+1)\pi}{z^2+(n+\frac{1}{2})^2\pi^2} = \frac{(-1)^n(2n+1)\pi}{\left(z-(n+\frac{1}{2})\pi i\right)\left(z+(n+\frac{1}{2})\pi i\right)}$$

$$=(-i)(-1)^n\left[\frac{1}{z-(n+\frac{1}{2})\pi i} - \frac{1}{z+(n+\frac{1}{2})\pi i}\right]$$

$$= \frac{(-1)^{n+1}i}{z-(n+\frac{1}{2})\pi i} - \frac{(-1)^{n+1}i}{z+(n+\frac{1}{2})\pi i}$$

So, for [redacted] poles at $\frac{1}{2}\pi i, \frac{3}{2}\pi i, \ldots$ it is the [redacted] residue as in part a).

For poles at $-\frac{1}{2}\pi i, -\frac{3}{2}\pi i, \ldots$, write $-(n+\frac{1}{2}) = n'+\frac{1}{2}$ with n' negative. So $n=-1-n'$, $n=0,1,\ldots, \Rightarrow n'=-1,-2,-3,\ldots$

$$-\frac{(-1)^{n+1}i}{z+(n+\frac{1}{2})\pi i} = -\frac{(-1)^{-n'}i}{z-(n'+\frac{1}{2})\pi i} = \frac{(-1)^{n'+1}i}{z-(n'+\frac{1}{2})\pi i}$$

So even for poles with negative imaginary parts, the residues are the same as in part a).

c) For $\omega > 0$

$$I(\omega) = \oint_C e^{i\omega z}\,\mathrm{sech}\, z\, dz$$

$$= 2\pi i \sum_{n=0}^{\infty} e^{i\omega z_n}(-1)^{n+1} i$$

(Only considering poles in the upper half plane)

$$= 2\pi \sum_{n=0}^{\infty} (-1)^n e^{i\omega(n+\frac{1}{2})\pi i}$$

$$= 2\pi \sum_{n=0}^{\infty} (-1)^n e^{-(n+\frac{1}{2})\pi\omega} = 2\pi e^{-\frac{\pi\omega}{2}} \left[1 - e^{-\pi\omega} + e^{-2\pi\omega} - e^{-3\pi\omega} + \cdots\right]$$

$$= 2\pi e^{-\pi\omega/2} \Big/ \left(1 + e^{-\pi\omega}\right) = \frac{2\pi}{e^{\pi\omega/2} + e^{-\pi\omega/2}} = \frac{\pi}{\cosh\frac{\pi\omega}{2}} = \pi\,\mathrm{sech}\,\frac{\pi\omega}{2}$$

Since $I(\omega)$ is an even function of ω.

$$\boxed{\hat{f}(\omega) = \pi\,\mathrm{sech}\,\frac{\pi\omega}{2}} \quad \text{for all } \omega \in \mathbb{R}.$$

2. a) For $i = 1$

$$\frac{d}{d\lambda}\left(g_{11}\frac{dx^1}{d\lambda}\right) - \frac{1}{2}\frac{\partial g_{kl}}{\partial x^1}\frac{dx^k}{d\lambda}\frac{dx^l}{d\lambda} = 0$$

$$\boxed{\frac{d}{d\lambda}\left(\frac{1}{(x^2)^2}\frac{dx^1}{d\lambda}\right) = 0} \xrightarrow[\text{to}]{\text{equivalent}} \boxed{\frac{d^2x^1}{d\lambda^2} - \frac{1}{x^2}\frac{dx^1}{d\lambda}\frac{dx^2}{d\lambda} = 0}$$

For $i =$ [redacted] 2

$$\frac{d}{d\lambda}\left(g_{22}\frac{dx^2}{d\lambda}\right) \text{[redacted]} - \frac{1}{2}\left\{\frac{\partial g_{11}}{\partial x^2}\left(\frac{dx^1}{d\lambda}\right)^2 + \frac{\partial g_{22}}{\partial x^2}\left(\frac{dx^2}{d\lambda}\right)^2\right\} = 0$$

$$\boxed{\frac{d}{d\lambda}\left(\frac{1}{(x^2)^2}\frac{dx^2}{d\lambda}\right) + \frac{1}{(x^2)^3}\left\{\left(\frac{dx^1}{d\lambda}\right)^2 + \left(\frac{dx^2}{d\lambda}\right)^2\right\} = 0}$$

equivalent to $\boxed{\frac{d^2x^2}{d\lambda^2} + \frac{1}{x^2}\left\{\left(\frac{dx^1}{d\lambda}\right)^2 - \left(\frac{dx^2}{d\lambda}\right)^2\right\} = 0}$

b) $\frac{1}{(x^2)^2}\frac{dx^1}{d\lambda} = \frac{1}{R} \Rightarrow \frac{dx^1}{d\lambda} = \frac{(x^2)^2}{R}$

$$\frac{1}{(x^2)^2}\left[\left(\frac{dx^1}{d\lambda}\right)^2+\left(\frac{dx^2}{d\lambda}\right)^2\right] = \frac{1}{(x^2)^2}\left(\frac{dx^1}{d\lambda}\right)^2\left[1+\left(\frac{dx^2}{dx^1}\right)^2\right]$$

$$= \frac{1}{(x^2)^2}\frac{(x^2)^4}{R^2}\left[1+\left(\frac{dx^2}{dx^1}\right)^2\right] = \frac{(x^2)^2}{R^2}\left[1+\left(\frac{dx^2}{dx^1}\right)^2\right]$$

So $\boxed{\left(\frac{x^2}{R}\right)^2\left[\left(\frac{dx^2}{dx^1}\right)^2+1\right] = 1}$

is our ODE.

c) $$\left(\frac{dx^2}{dx^1}\right)^2 = \frac{R^2}{(x^2)^2} - 1$$

or $$\frac{dx^2}{dx^1} = \pm\frac{\sqrt{R^2-(x^2)^2}}{x^2}$$

or $$\pm\int\frac{x^2\,dx^2}{\sqrt{R^2-(x^2)^2}} = \int dx^1$$

$$\pm\sqrt{R^2-(x^2)^2} = x^1 + \text{constant}$$

Write this as $\pm\sqrt{R^2-(x^2)^2} = x^1 - a$

$$\Rightarrow \quad (x^1-a)^2 = R^2-(x^2)^2$$

or $\boxed{(x^1-a)^2+(x^2)^2 = R^2}$

(a,0)

The geodesics are half circles centered around $(a,0)$. Of course $x^2=0$ is not included in the space.

3. a) $\frac{d}{dt}u_2(t) = -u_2(t) \Rightarrow u_2(t) = u_2(0)e^{-t} = e^{-t}$

$\frac{d}{dt}u_1(t) = -u_1(t) + e^{-t}$

$\frac{d}{dt}u_1(t) + u_1(t) = e^{-t}$ Using integrating factor

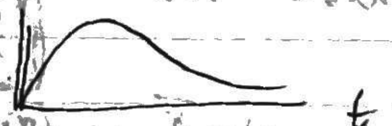
$e^{-t}\frac{d}{dt}\left(e^{t}u_1(t)\right) = e^{-t}$

So $\frac{d}{dt}\left(e^{t}u_1(t)\right) = 1$

Integrating from zero $e^{t}u_1(t) = t$, since $u_1(0)=0$

$u_1(t) = te^{-t}$.

So $\boxed{\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} te^{-t} \\ e^{-t} \end{pmatrix}}$



b) $u_1(t) = te^{-t}$

$0 = \frac{d}{dt}u_1(t) = e^{-t} - te^{-t} \Rightarrow t=1$

$u_1''(t) = -e^{-t} - e^{-t} + te^{-t} = (t-2)e^{-t}$ $u_1''(1) < 0$

So $u_1(t)$ is maximized at $t=1$. $\boxed{u_1^{max} = u_1(1) = e^{-1}}$

c) As $t \to \infty$, $e^{-t} \to 0$ and $te^{-t} \to 0$.

So $\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

4. a) $$\int_0^L dx \sin\frac{\pi x}{L} = \left[\frac{-\cos\frac{\pi x}{L}}{\frac{\pi}{L}}\right]_0^L = \frac{L}{\pi}\left[-\cos\pi + \cos 0\right]$$
$$= \frac{2L}{\pi}$$

So $$\int_{Box} \rho(x,y,z,0)\, dx\, dy\, dz = \left(\frac{2L}{\pi}\right)^3$$

$$N_0 = A\left(\frac{2L}{\pi}\right)^3 \Rightarrow \boxed{A = \frac{\pi^3 N_0}{8L^3}}$$

In general, $\rho(x,y,z,t)$ would be expanded in terms of eigenfunctions of the Laplacian. These are [redacted]

$u_{lmn}(x,y,z) = \sqrt{\frac{8}{L^3}} \sin\frac{l\pi x}{L} \sin\frac{m\pi y}{L} \sin\frac{n\pi z}{L}$, [redacted] $l = 1, 2, \ldots$, $m = 1, 2, \ldots$, $n = 1, 2, \ldots$

with the right boundary condition

$\vec{\nabla}^2 u_{lmn} = -$ [redacted] $\frac{\pi^2}{L^2}(l^2 + m^2 + n^2)\, u_{lmn} = \lambda_{lmn} u_{lmn}$

If $\rho(x,y,z,t) = \sum_{lmn} C_{lmn}(t)\, u_{lmn}(x,y,z)$

then $\frac{d}{dt} C_{lmn}(t) = D \lambda_{lmn}$ [redacted] $C_{lmn}(t)$

$\Rightarrow C_{lmn}(t) = e^{D\lambda_{lmn} t}\, C_{lmn}(0)$

For us, the initial [redacted] ρ is proportional to $u_{111}(x,y,z) \Rightarrow$ Only nontrivial coefficient is C_{111}.

So $\rho(x,y,z,t) = e^{\lambda_{111} t}\, \rho(x,y,z,t)$

$\lambda_{111} = -\frac{3\pi^2}{L^2}$

$$\Rightarrow \boxed{\rho(x,y,z,t) = \frac{\pi^3 N_0}{8L^3} \sin\frac{\pi x}{L} \sin\frac{\pi y}{L} \sin\frac{\pi z}{L}\, e^{-\frac{3\pi^2 D}{L^2} t}}$$

b) The density decays as $e^{-\frac{3\pi^2 D}{L^2} t}$.

So, for the particle number to drop by $\frac{1}{2}$.

We need $\frac{1}{2} = e^{-\frac{3\pi^2 D}{L^2} t}$

So $$\boxed{t_{\frac{1}{2}} = \frac{L^2}{3\pi^2 D} \ln 2}$$