Sol: Probset 4

1. (Vaughn 3.4) We want to show that $\mathcal{L}_v(d\sigma) = d(\mathcal{L}_v\sigma)$ for an arbitrary p form σ.

We of course assume σ is smooth, thereby guarteeing continuous mixed derivatives. That implies order of mixed partial derivatives do not matter.

We will also assume Leibnitz rule for Lie derivatives. That works for tensor products, wedge products as well as for contractions.

Proof by induction: $\mathcal{L}_v(d\sigma) = d(\mathcal{L}_v\sigma)$

Case $p=0$: When σ is a zero form, it is just a function.

$$d\sigma = \frac{\partial \sigma}{\partial x^i} dx^i$$

$$\mathcal{L}_v(d\sigma) = \left(v^j \frac{\partial}{\partial x^j}\frac{\partial \sigma}{\partial x^i} + \frac{\partial \sigma}{\partial x^j}\frac{\partial v^j}{\partial x^i}\right) dx^i$$ (Lie deriv for 1-form)

$$\mathcal{L}_v(\sigma) = v^j \frac{\partial \sigma}{\partial x^j},$$

So $$d\mathcal{L}_v(\sigma) = \frac{\partial}{\partial x^i}\left(v^j \frac{\partial \sigma}{\partial x^j}\right) dx^i$$

$$= \left(v^j \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\sigma + \frac{\partial \sigma}{\partial x^j}\frac{\partial v^j}{\partial x^i}\right) dx^i = \mathcal{L}_v(d\sigma)$$

since $\dfrac{\partial^2 \sigma}{\partial x^i \partial x^j} = \dfrac{\partial^2 \sigma}{\partial x^j \partial x^i}$

Now assume the statement to be true for $p = 0, \dots, k$

Case $p = k+1$:

$$\sigma = \frac{1}{(k+1)!}\, \sigma_{i_1 \cdots i_{k+1}}\, dx^{i_1} \wedge \cdots \wedge dx^{i_{k+1}}$$

$$d\sigma = \frac{1}{(k+1)!}\, d(\sigma_{i_1 \cdots i_{k+1}}) \wedge (dx^{i_1} \wedge \cdots \wedge dx^{i_{k+1}})$$

Notice that $\underbrace{d(x^{i_1} dx^{i_2} \wedge \cdots \wedge dx^{i_{k+1}})}_{d \text{ of a } k \text{ form}} = dx^{i_1} \wedge \cdots \wedge dx^{i_{k+1}}$

$$\mathcal{L}_V(d\sigma) = \frac{1}{(k+1)!} \Big[\mathcal{L}_V (d\sigma_{i_1 \cdots i_{k+1}}) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{k+1}}$$
$$+ d\sigma_{i_1 \cdots i_{k+1}} \wedge \mathcal{L}_V (dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_{k+1}}) \Big]$$

$$d\mathcal{L}_V(\sigma) = \frac{1}{(k+1)!}\, d\Big[\mathcal{L}_V(\sigma_{i_1 \cdots i_{k+1}})\, dx^{i_1} \wedge \cdots \wedge dx^{i_{k+1}}$$
$$+ \sigma_{i_1 \cdots i_{k+1}}\, \mathcal{L}_V (dx^{i_1} \wedge \cdots \wedge dx^{i_{k+1}}) \Big]$$
$$= \frac{1}{(k+1)!} \Big[d\mathcal{L}_V(\sigma_{i_1 \cdots i_{k+1}}) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{k+1}}$$
$$+ d(\sigma_{i_1 \cdots i_{k+1}}) \wedge \mathcal{L}_V (dx^{i_1} \wedge \cdots \wedge dx^{i_{k+1}})$$
$$+ \sigma_{i_1 \cdots i_{k+1}}\, d\mathcal{L}_V (dx^{i_1} \wedge \cdots \wedge dx^{i_{k+1}}) \Big]$$

Since $x^{i_1} dx^{i_2} \wedge \cdots \wedge dx^{i_{k+1}}$ is a k form
$\mathcal{L}_V(d(x^{i_1} dx^{i_2} \wedge \cdots \wedge dx^{i_{k+1}})) = d\mathcal{L}_V(x^{i_1} dx^{i_2} \wedge \cdots \wedge dx^{i_{k+1}})$

So, $d\mathcal{L}_V(dx^{i_1} \wedge \cdots \wedge dx^{i_{k+1}})$

$= d\mathcal{L}_V(d(x^{i_1} dx^{i_2} \wedge \cdots \wedge dx^{i_{k+1}}))$

$= d^2\mathcal{L}_V(x^{i_1} dx^{i_2} \wedge \cdots \wedge dx^{i_{k+1}}) = 0$

[redacted]

$$\text{So } d\mathcal{L}_V(\sigma) = \frac{1}{(k+1)!}\left[d\mathcal{L}_V(\sigma_{i_1 \cdots i_{k+1}}) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{k+1}} + d(\sigma_{i_1 \cdots i_{k+1}}) \wedge \mathcal{L}_V(dx^{i_1} \wedge \cdots \wedge x^{i_{k+1}})\right]$$

Since $\sigma_{i_1 \cdots i_{k+1}}$ is a 0-form

$d\mathcal{L}_V(\sigma_{i_1 \cdots i_{k+1}}) = \mathcal{L}_V(d\sigma_{i_1 \cdots i_{k+1}})$

Thus $\mathcal{L}_V(d\sigma) = d(\mathcal{L}_V \sigma)$

QED

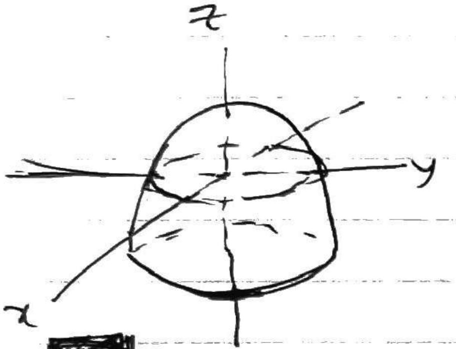
2. (Vaughn 3.9)

$x = \xi\eta\cos\phi \qquad y = \xi\eta\sin\phi$

$z = \frac{1}{2}(\xi^2 - \eta^2)$

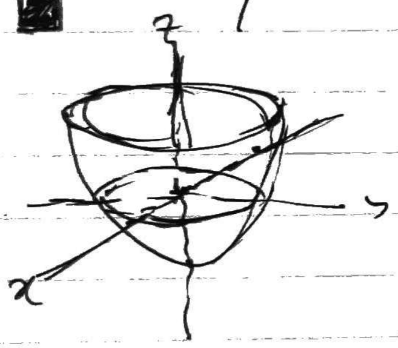
(i) ξ constant $\Rightarrow z = \frac{1}{2}(\xi^2 - \eta^2)$

$$= \frac{1}{2}\left[\xi^2 - \frac{1}{\xi^2}(x^2+y^2)\right]$$



Inverted paraboloid of rotation

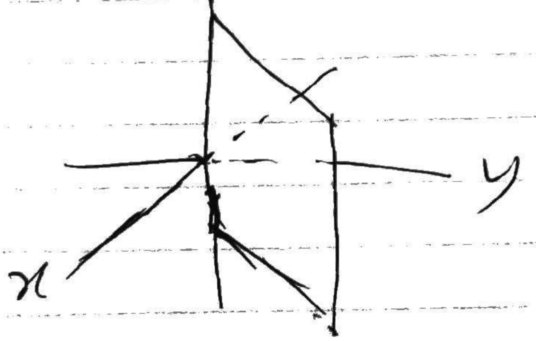
[redacted] η constant $\Rightarrow z = \frac{1}{2}\left[\frac{1}{\eta^2}(x^2+y^2) - \eta^2\right]$



'Right side up' paraboloids of rotation.

[redacted] ϕ constant $\Rightarrow \frac{x}{y} = \tan\phi$

We assume $\xi, \eta \geq 0$. Half planes.



ii) Informally, small arclength $ds^2 = dx^2 + dy^2 + dz^2$

$$= (d(\xi\eta)\cos\phi - \xi\eta\sin\phi\, d\phi)^2 + (d(\xi\eta)\sin\phi + \xi\eta\cos\phi\, d\phi)^2$$
$$+ (\xi d\xi - \eta d\eta)^2$$

$$= (d(\xi\eta))^2(\cos^2\phi + \sin^2\phi) + \xi^2\eta^2(\sin^2\phi + \cos^2\phi)\,d\phi^2$$
$$+ (\xi^2 d\xi^2 + \eta^2 d\eta^2 - 2\xi\eta\, d\xi\, d\eta)$$
$$= (\xi\, d\eta + \eta\, d\xi)^2 + \xi^2\eta^2\, d\phi^2$$
$$+ (\xi^2 d\xi^2 + \eta^2 d\eta^2 - 2\xi\eta\, d\xi\, d\eta)$$
$$= (\xi^2+\eta^2)(d\xi^2 + d\eta^2) + \xi^2\eta^2\, d\phi^2$$

$$[g] = \begin{pmatrix} \xi^2+\eta^2 & & \\ & \xi^2+\eta^2 & \\ & & \xi^2\eta^2 \end{pmatrix} \quad \text{Metric Tensor}$$

g_{ij} element

With $i,j = 1, 2, 3 \Rightarrow \xi, \eta, \phi$

$$[\bar{g}] = \begin{pmatrix} (\xi^2+\eta^2)^{-1} & & \\ & (\xi^2+\eta^2)^{-1} & \\ & & \xi^{-2}\eta^{-2} \end{pmatrix}$$

g^{ij} elements

Natural Volume form $\quad dx\wedge dy\wedge dz = \sqrt{\det g}\; d\xi\wedge d\eta\wedge d\phi$

$$= \sqrt{(\xi^2+\eta^2)^2\xi^2\eta^2}\; d\xi\wedge d\eta\wedge d\phi$$
$$= (\xi^2+\eta^2)\xi\eta\; d\xi\wedge d\eta\wedge d\phi$$

(iii) Laplacian

$$\Delta = \frac{1}{\sqrt{\det g}}\,\partial_i \sqrt{\det g}\; g^{ij}\,\partial_j$$

$$= \frac{1}{\xi\eta(\xi^2+\eta^2)}\left[\frac{\partial}{\partial\xi}\,\frac{\xi\eta\cancel{(\xi^2+\eta^2)}}{\cancel{(\xi^2+\eta^2)}}\,\frac{\partial}{\partial\xi}\right.$$

$$\left. + \frac{\partial}{\partial\eta}\,\frac{\xi\eta\cancel{(\xi^2+\eta^2)}}{\cancel{(\xi^2+\eta^2)}}\,\frac{\partial}{\partial\eta} + \frac{\partial}{\partial\phi}\,\frac{\xi\eta(\xi^2+\eta^2)}{\xi^2\eta^2}\,\frac{\partial}{\partial\phi}\right]$$

$$= \frac{1}{\xi^2+\eta^2}\left(\frac{\partial}{\partial\xi}\xi\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta}\eta\frac{\partial}{\partial\eta}\right)$$

$$+ \frac{1}{\xi^2\eta^2}\,\frac{\partial^2}{\partial\phi^2}$$

3. (Vaughn 3.19) We have a slight problem here!
Vaughn defines $\sigma\wedge\tau = \frac{1}{2}(\sigma\otimes\tau - \tau\otimes\sigma)$ for 1 forms σ, τ. This is not consistent with his convention for contraction.
We will work with $\sigma\wedge\tau = \sigma\otimes\tau - \tau\otimes\sigma$.

If you get extra factors of $\frac{1}{2}$ thank's to the book's slightly inconsistent conventions, no penalty!

So $\omega = dp_k \wedge dq^k = dp_k \otimes dq^k - dq^k \otimes dp_k$

If $X_f = u^k \frac{\partial}{\partial q^k} + v_k \frac{\partial}{\partial p_k}$

$$i_{X_f}\omega = \left(dp_k \otimes dq^k - dq^k \otimes dp_k\right)(X_f, \cdot)$$

$$= v_k \, dq^k - u^k \, dp_k$$

Since $i_{X_f}\omega = -df = -\frac{\partial f}{\partial q^k} dq^k - \frac{\partial f}{\partial p_k} dp_k$

We can read off $u_k = \frac{\partial f}{\partial p_k}$, $v_k = -\frac{\partial f}{\partial q^k}$

$$\Rightarrow X_f = \frac{\partial f}{\partial p_k}\frac{\partial}{\partial q^k} - \frac{\partial f}{\partial q^k}\frac{\partial}{\partial p_k}$$

ii) $\{f, g\} = \omega(X_f, X_g)$

$$= \left(dp_k \otimes dq^k - dq^k \otimes dp^k\right)(X_f, X_g)$$

$$= \left(-\frac{\partial f}{\partial q^k}\right)\frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k}\left(-\frac{\partial g}{\partial q^k}\right)$$

$$= \frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q^k} - \frac{\partial f}{\partial q^k}\frac{\partial g}{\partial p_k}$$

To show $[X_f, X_g] = X_{\{f,g\}}$

One can directly compute the commutator and reorganize it as $X_{\{f,g\}}$. Full credit for that.

Alternatively, we start by noticing
$$X_f\, g = \{f, g\}.$$

Then use $\mathcal{L}_v(\sigma) = d(i_v\sigma) + i_v(d\sigma)$ [Vaugh page 123]

[redacted] with $\sigma = \omega$ and $v = X_f$

$$\mathcal{L}_{X_f}(\omega) = d(i_{X_f}\omega) + i_{X_f}(d\omega)$$

$$d\omega = d(dp_k \wedge dq^k) = 0 \qquad i_{X_f}\omega = -df$$

$$d(i_{X_f}\omega) = -d^2 f = 0$$

So $\mathcal{L}_{X_f}(\omega) = 0$ [symplectic form invariant under canonical transform$^{\underline{n}}$.]

Now we apply the Leibniz rule to contraction

$$\mathcal{L}_{X_f}(i_{X_g}\omega) = i_{\mathcal{L}_{X_f}(X_g)}\,\omega + i_{X_g}(\mathcal{L}_{X_f}\omega)$$

$\mathcal{L}_{X_f}(\omega) = 0$ as shown above

$\mathcal{L}_{X_f}(X_g) = [X_f, X_g]$ from Lie deriv of vector fields

So $\mathcal{L}_{X_f}(i_{X_g}\omega) = i_{[X_f, X_g]}\,\omega$

But $\mathcal{L}_{X_f}(i_{X_g}\omega) = \mathcal{L}_{X_f}(-dg) = -d\,\mathcal{L}_{X_f}(g)$

$$= -d\{f, g\}$$

So $i_{[X_f, X_g]}\,\omega = -d\{f, g\} = i_{X_{\{f,g\}}}\,\omega$

Then, by ~~[illegible]~~ nondegeneracy of ω

$$[X_f, X_g] = X_{\{f,g\}} \;\blacksquare \qquad \text{QED.}$$

iii) $\frac{df}{dt} = \dot{q}^k$ [redacted] $\frac{\partial f}{\partial q^k} + \dot{p}_k \frac{\partial f}{\partial p_k} + \frac{\partial f}{\partial t} = \frac{\partial H}{\partial p_k}\frac{\partial f}{\partial q^k} - \frac{\partial H}{\partial q^k}\frac{\partial f}{\partial p_k} + \frac{\partial f}{\partial t}$

$$= \{H, f\} + \frac{\partial f}{\partial t}$$

4. (Vaughn [redacted] 4.8)

Near $z = m\pi$, $m \in \mathbb{Z}$

$$\frac{1}{\sin^2 z} = \frac{1}{\sin^2(z - m\pi)} = \frac{1}{(z-m\pi)^2} + \text{regular analytic function}$$

Define $f_0 : \mathbb{C} - \{m\pi + i0 \mid m \in \mathbb{Z}\} \to \mathbb{C}$

$$f_0(z) = \frac{1}{\sin^2 z} - \sum_{n=-\infty}^{\infty} \frac{1}{(z+n\pi)^2}$$

Each part is well defined [redacted] in the domain

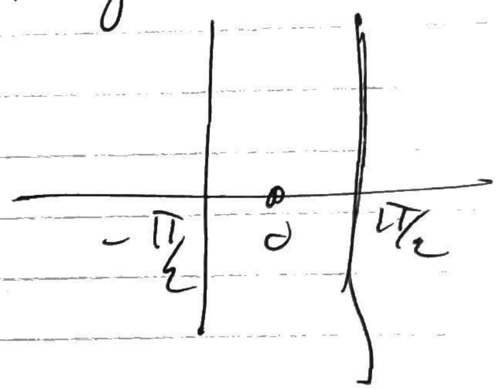
Note that as $z \to m\pi$, $f_0(z)$ has no singularity. So f_0 can be extended to an entire function $f : \mathbb{C} \to \mathbb{C}$.

Also see that $f(z+\pi) = f(z)$. It is a periodic function.

Our Strategy would be to show that f is bounded. Then Liouville's theorem

says it is a constant function.
We will also show that $f(z) \to 0$ as $\text{im}(z) \to \pm\infty$. Hence the constant must be zero. If we show $f(z) = 0$ we have proven our identity.
Since $f(z+\pi) = f(z)$ we could restrict ourselves to the region

$$-\frac{\pi}{2} \le \text{re}(z) \le \pi/2$$



Let $z = x+iy$
$x, y \in \mathbb{R}$.

$$\sin z = \frac{e^{ix}e^{-y} - e^{-ix}e^{y}}{2i}$$

$$|\sin^2 z| = \frac{e^{2y} + e^{-2y} - 2\cos 2x}{4}$$

as $|y| \to \infty$ $|\sin^2 z| \to \infty$
implying $\frac{1}{\sin^2 z} \to 0$

Also $$\left|\sum_{n=-\infty}^{\infty} \frac{1}{(z+n\pi)^2}\right| \le \sum_{n=-\infty}^{\infty} \left|\frac{1}{(z+n\pi)^2}\right|$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{(x+n\pi)^2+y^2} = \frac{1}{x^2+y^2} + \sum_{n'=-\infty}^{-1} \frac{1}{(x+n'\pi)^2+y^2} + \sum_{n=1}^{\infty} \frac{1}{(x+n\pi)^2+y^2}$$

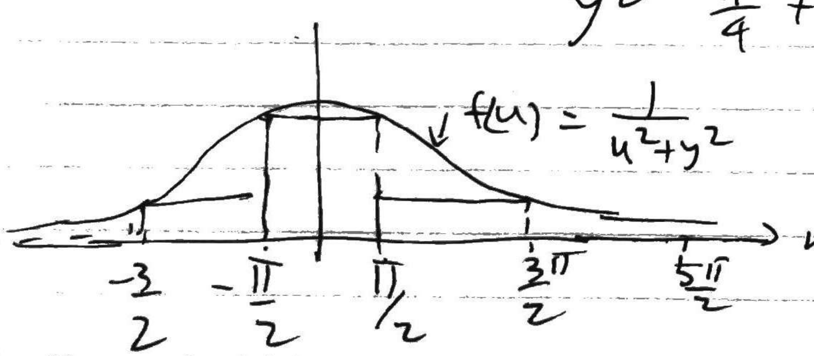
Putting $n' = -n$

$$\sum_{n=-\infty}^{\infty} \frac{1}{(x+n\pi)^2+y^2} = \frac{1}{x^2+y^2} + \sum_{n=1}^{\infty} \frac{1}{(n\pi-x)^2+y^2} + \sum_{n=1}^{\infty} \frac{1}{(n\pi+x)^2+y^2}$$

Since $|x| \leq \frac{\pi}{2}$

$$\sum_{n=-\infty}^{\infty} \frac{1}{(x+n\pi)^2+y^2} \leq \frac{1}{y^2} + 2\sum_{n=1}^{\infty} \frac{1}{(n-\frac{1}{2})^2\pi^2+y^2}$$

$$= \frac{1}{y^2} + \frac{1}{\frac{\pi^2}{4}+y^2} + \sum_{m=-\infty}^{\infty} \frac{1}{(2m+\frac{1}{2})^2\pi^2+y^2}$$



$$\leq \frac{1}{y^2} + \frac{1}{\frac{\pi^2}{4}+y^2} + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{dx}{x^2+y^2}$$

$$= \frac{1}{y^2} + \frac{1}{\frac{\pi^2}{4}+y^2} + \frac{1}{|y|}$$

As $|y| \to \infty$ The upper bound $\to 0$

Since $f(z)$ is continuous, in any bounded region, it is bounded.

Since it tends to zero as $|y| \to \infty$

It is bounded everywhere.

Hence $f(z) = $ constant and the constant is zero.

$$\frac{1}{\sin^2 z} = \sum_{n=-\infty}^{\infty} \frac{1}{(z+n\pi)^2}$$

$$\frac{1}{\sin^2 z} - \frac{1}{z^2} = \sum_{n=1}^{\infty} \left[\frac{1}{(n\pi+z)^2} + \frac{1}{(n\pi-z)^2} \right]$$

$$\sin z = z\left(1 - \frac{z^2}{6} + O(z^4)\right)$$

$$\frac{1}{\sin^2 z} = \frac{1}{z^2}\left(1 + \frac{z^2}{3} + O(z^4)\right) \Rightarrow \frac{1}{\sin^2 z} - \frac{1}{z^2} = \frac{1}{3} + O(z^2)$$

Letting $z \to 0$ on the RHS

$$\frac{1}{3} = 2\sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\sum_4(z) = \sum_{n=1}^{\infty} \frac{1}{(z+n\pi)^4}$$

$$\frac{d}{dz}\left(\frac{1}{\sin^2 z}\right) = -\frac{2\cos z}{\sin^3 z}$$

$$\frac{d^2}{dz^2}\left(\frac{1}{\sin^2 z}\right) = -\frac{2(-\sin z)}{\sin^3 z} + \frac{3\cdot 2\cos^2 z}{\sin^4 z}$$

$$= \frac{2}{\sin^2 z} + 6\frac{\cos^2 z}{\sin^4 z}$$

$$\frac{d^2}{dz^2}\sum_{n=-\infty}^{\infty}\frac{1}{(z+n\pi)^2} = 6\sum_{n=-\infty}^{\infty}\frac{1}{(z+n\pi)^4} = 6\sum_4(z)$$

So $$\sum_4(z) = \frac{\cos^2 z}{\sin^4 z} + \frac{1}{3}\frac{1}{\sin^2 z}$$

$$= \frac{1-\sin^2 z}{\sin^4 z} + \frac{1}{3}\frac{1}{\sin^2 z}$$

$$= \frac{1}{\sin^4 z} - \frac{2}{3}\frac{1}{\sin^2 z}$$

Now $$\frac{1}{\sin^2 z} = \left(\frac{1}{z - \frac{z^3}{6} + \frac{z^5}{120} + O(z^7)}\right)^2$$

$$= \frac{1}{z^2}\left(1 - \frac{z^2}{6} + \frac{z^4}{120} + O(z^6)\right)^{-2}$$

$$= \frac{1}{z^2}\left(1 + \frac{z^2}{3} + \frac{z^4}{15} + O(z^6)\right)$$

$$\sum_{n=-\infty}^{\infty} \frac{1}{(z+n\pi)^4}$$

$$\frac{1}{6}\frac{d^2}{dz^2}\frac{1}{\sin^2 z} = \frac{1}{6}\frac{d^2}{dz^2}\left[\frac{1}{z^2}+\frac{1}{3}+\frac{z^2}{15}+O(z^4)\right]$$

$$= \frac{1}{6}\left[\frac{6}{z^4}+\frac{2}{15}+O(z^2)\right]$$

$$= \frac{1}{z^4}+\frac{1}{45}+O(z^2)$$

$\Rightarrow$

$$\frac{1}{z^4}+\frac{1}{45}+O(z^2) = \frac{1}{z^4}+\sum_{n=1}^{\infty}\left[\frac{1}{(n\pi+z)^4}+\frac{1}{(n\pi-z)^4}\right]$$

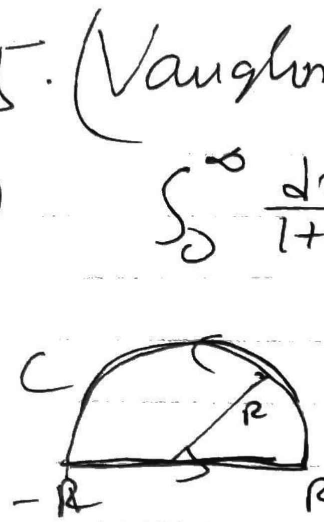
Cancel $\frac{1}{z^4}$ from both side and then let $z \to 0$

$$\frac{1}{45} = 2\sum_{n=1}^{\infty}\frac{1}{(n\pi)^4}$$

$$\Rightarrow \zeta(4) = \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

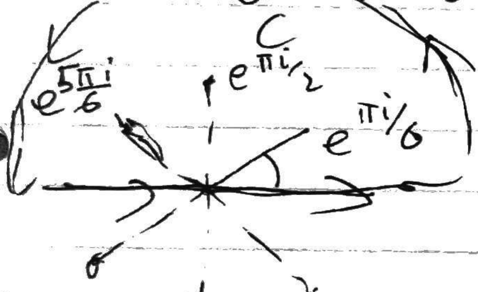
5. (Vaughn 4.13)

(i) $\int_0^\infty \frac{dx}{1+x^6} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{dx}{1+x^6} = \lim_{R\to\infty}\int_{-R}^{R}\frac{dx}{1+x^6}$



We can add the half circle $z=Re^{i\theta}$, $0\le\theta\le\pi$ since the contribution of the arc $\sim R^{-5}$.

$\frac{1}{2}\oint_C \frac{dz}{1+z^6}$



$z^6 = e^{i\pi}$

Say z_0 is a root. $z_0^6=-1$

$z^6+1 \simeq 6z_0^5(z-z_0)$

So the residue is

$\frac{1}{6z_0^5} = \frac{z_0}{6z_0^6} = -\frac{z_0}{6}$

Only roots in the upper half place contribute!

$\frac{1}{2}\oint_C \frac{dz}{1+z^6} = \frac{1}{\cancel{2}}\cancel{2}\pi i\left(-\frac{1}{6}\right)\left[e^{\pi i/6}+i+e^{5\pi i/6}\right]$

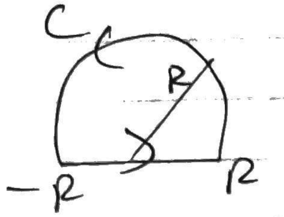
$= -\frac{\pi i}{6}\left[i+\cancel{\cos\frac{\pi}{6}}+i\sin\frac{\pi}{6}-\cancel{\cos\frac{\pi}{6}}+i\sin\frac{\pi}{6}\right]$

$= \frac{\pi}{6}\left[1+2\sin\frac{\pi}{6}\right] = \frac{\pi}{6}\left[1+2\cdot\frac{1}{2}\right]$

$= \frac{\pi}{3}$

So $\int_0^\infty \frac{dx}{1+x^6} = \frac{\pi}{3}$

(ii) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)(1-2\alpha x+x^2)} = \oint_C \frac{dz}{(z^2+1)(z^2-2\alpha z+1)}$



Once more, the extra arc's contr. $\sim \frac{1}{R^3}$

$$z^2-2\alpha z+1 = (z-\alpha)^2+(1-\alpha^2)$$

We note that for real α, with $|\alpha|<1$ this integral is well defined. In fact for $|\alpha|<1$, complex α, one could expand in α. It is analytic in α there. We will start from there and consider other α later by continuation.

$$z^2-2\alpha z+1=0 \Rightarrow z=\alpha_\pm = \alpha \pm\sqrt{\alpha^2-1}$$

So that for $|\alpha|\ll 1$ $\quad \alpha_+ = i\sqrt{1-\alpha^2}+\alpha$
$\simeq i+\alpha$

$\alpha_- = -i\sqrt{1-\alpha^2}-\alpha$
$\simeq -i-\alpha$

Only α_+ is in the upper half plane. Also $z^2+1=0$ has roots $z=\pm i$

Only $z=+i$ is in the upper half plane.

Near $z=i$ $\qquad z^2+1 = (z+i)(z-i) \approx 2i(z-i)$

$$z^2-2\alpha z+1 \simeq -2\alpha i \quad \text{since } z^2+1\simeq 0$$

Near $z=\alpha_+$ [redacted] $z^2+1 \simeq 2\alpha z \simeq 2\alpha\alpha_+$

$$z^2-2\alpha+1 = (z-\alpha_+)(z-\alpha_-)$$

$$\approx (\alpha_+-\alpha_-)(z-\alpha_+) = 2\sqrt{\alpha^2-1}\,(z-\alpha_+)$$

So $\displaystyle\oint_C \frac{dz}{(z^2+1)(z^2-2\alpha z+1)} = 2\pi i\left[\frac{1}{2i(-2i\alpha)} + \frac{1}{2\alpha\alpha_+\, 2\sqrt{\alpha^2-1}}\right]$ [redacted]

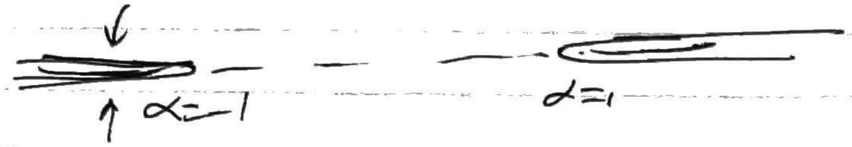
$$= \frac{\pi i}{2\alpha}\left[1+\frac{1}{\sqrt{\alpha^2-1}\,\alpha_+}\right] = \frac{\pi i}{2\alpha}\left[1+\frac{\alpha_-}{\text{[redacted]}\sqrt{\alpha^2-1}\,\text{[redacted]}\,\alpha_+\alpha_-}\right]$$

$$= \frac{\pi i}{2\alpha}\left[1+\frac{\alpha-\sqrt{\alpha^2-1}}{\sqrt{\alpha^2-1}}\right] \qquad \left(\text{Since } \alpha_+\alpha_-=1\right)$$

$$= \frac{\pi i}{2\alpha}\,\frac{\alpha}{\sqrt{\alpha^2-1}} = \frac{\pi}{2\sqrt{1-\alpha^2}}$$ [redacted]

Note that this is analytic for $|\alpha|<1$

For $|\alpha|>1$ there are cuts on the real axis where coming from above or below the real axis makes a difference.



$$\int_{-\infty}^{\infty}\frac{dx}{(1+x^2)(1-2\alpha x+x^2)}=\frac{\pi}{2\sqrt{1-\alpha^2}}$$

6. (Vaughn 5.7) $\quad \frac{d^2u}{dz^2}+\left(\lambda-c\frac{e^{-\alpha z}}{z}\right)u(z)=0$

(i) Clearly $z=0$ is singular point

If we write $\quad \frac{d^2u(z)}{dz^2}+p(z)\frac{du(z)}{dz}+q(z)\,u(z)=0$

$P(z)=zp(z)=0 \qquad Q(z)=z^2q(z)=\lambda z^2-cze^{-\alpha z}$ is analytic at $z=0$

so it is a regular point

Extra credit of 5. Look at $z=\infty$

use $z=\frac{1}{\omega}$, $u(z)=v(\omega)$, $\frac{d^2}{dz^2}=\left(\omega^2\frac{d}{d\omega}\right)^2=\omega^4\frac{d^2}{d\omega^2}+2\omega^3\frac{d}{d\omega}$

$\Rightarrow \left[\frac{d^2}{d\omega^2}+\frac{2}{\omega}\frac{d}{d\omega}+\frac{1}{\omega^4}\left(1-a\omega e^{-\alpha/\omega}\right)\right]v(\omega)=0 \Rightarrow$ Irregular singular pt.

As we said $0 = P(z) = \sum_{n=0}^{\infty} p_n z^n$

So $p_n = 0$, for all n

$$Q(z) = z^2 q(z) = -cz + (\lambda + \alpha c) z^2 - \frac{c\alpha^2}{2} z^3 + \frac{c\alpha^3}{6} z^4 + \cdots$$

$q_0 = 0$, $q_1 = -c$, $q_2 = \lambda + \alpha c$,

$q_3 = -\frac{c\alpha^2}{2}$, $q_4 = \frac{c\alpha^3}{6}$

[redacted] If we try $u(z) = \sum_{n=0}^{\infty} a_n z^n$

We get $n(n-1)a_n + \sum_{k=0}^{n} q_{n-k} a_k = 0$

$q_1 a_0 + q_0 a_1 = 0$ Since $q_0 = 0$ and $q_1 = -c$
(assume $c \neq 0$)

$\boxed{a_0 = 0}$ We set $a_1 = 1$

$2a_2 +$ [redacted] $q_1 a_1 + q_0 \cancel{a_2} = 0$

$2a_2 +$ [redacted] $c) = 0$

$\Rightarrow \boxed{a_2 = \frac{c}{2}}$

(21)

$$3\cdot 2\, a_3 + \cancel{q_3 a_0} + q_2 a_1 + q_1 a_2 + \cancel{q_0 a_3} = 0$$

$$6 a_3 + (\lambda + \alpha c) + (-c)\frac{c}{2} = 0$$

$$\Rightarrow a_3 = -\frac{1}{6}\left[\lambda + \alpha c - \frac{c^2}{2}\right]$$

So $u_1(z) = z + \frac{c}{2} z^2 - \frac{2\lambda + 2\alpha c - c^2}{12} z^3 + O(z^3)$

iii) Let's say $u_1(z) = z + a_2 z^2 + a_3 z^3 + \cdots$

To find the other solution, follow textbook procedure. Define $w(z)$ by $u_2(z) = w(z)\, u_1(z)$

$$w'(z) = \frac{A}{(u_1(z))^2} \qquad \text{Since } p(z) = 0$$

$$= \frac{A}{z^2}(1 + a_2 z + a_3 z^2 + \cdots)^{-2}$$

$$= \frac{A}{z^2}\left[1 - 2a_2 z + (3a_2^2 - 2a_3) z^2 + \cdots\right]$$

$$\Rightarrow w(z) = A\left[-\frac{1}{z} - 2a_2 \ln z + (3a_2^2 - 2a_3) z + \cdots\right]$$

$$\Rightarrow u_2(z) = -\frac{A}{z}\left[1 + 2a_2 z \ln z + (2a_3 - 3a_2^2) z^2 + \cdots\right] z\left[1 + a_2 z + a_3 z^2 + \cdots\right]$$

$$= -A\left[1 + a_2(2z\ln z + z) + 3(a_3 - a_2^2) z^2 + 2a_2^2 z^2 \ln z + \cdots\right]$$

Putting $A=-1$

$$u_2(z) = 1 + C\left(z\ln z + \frac{1}{2}\right) - \frac{1}{2}(\lambda + aC + C^2)z^2 + \frac{C^2}{2}z^2\ln z + o(z^2)$$

[Getting first two nontrivial terms of u_1, u_2 is good for full credit!]

7. (Vaughn 5.11)

(i) P_λ, Q_λ satisfy $u''(z) + \frac{2z}{z^2-1}u'(z) - \frac{\lambda(\lambda+1)}{z^2-1}u(z) = 0$

Eqn for Wronskian

$\Rightarrow W'(z) = -\frac{2z}{z^2-1}W(z)$

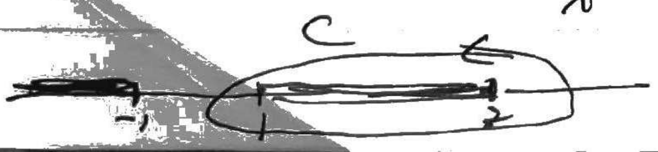
$\Rightarrow \frac{1}{W}\frac{dW}{dz} = -\frac{2z}{z^2-1} = \frac{d}{dt}\left[-\ln(z^2-1)\right]$

So $\ln W(z) = \underbrace{\ln A}_{\text{const}} - \ln(z^2-1)$

$W(z) = \frac{A}{z^2-1} \Rightarrow P_\lambda(z)Q'_\lambda(z) - P'_\lambda(z)Q_\lambda(z) = \frac{A}{z^2-1}$

ii) For large z, $Q_\lambda(z) \to \frac{[\Gamma(\lambda+1)]^2}{2\Gamma(2\lambda+2)}\left(\frac{2}{z}\right)^{\lambda+1}$ [See book Ex 5.18]

$P_\lambda(z)$ is tricky and not explicitly given in the book. $P_\lambda(z) = \frac{1}{2^\lambda 2\pi i}\oint_C \frac{(\xi^2-1)^\lambda}{(\xi-z)^{\lambda+1}}d\xi$ (5.133 modified)



$$P_\lambda(z) = -\frac{\sin\pi\lambda}{z^\lambda \pi} \int_1^z \frac{(\xi^2-1)^\lambda}{(z-\xi)^{\lambda+1}} d\xi$$ for large real z

$$= -\frac{\sin\pi\lambda}{z^\lambda \pi} z^\lambda \int_{1/z}^1 \frac{u^{2\lambda}\left(1-\frac{1}{z^2u^2}\right)^\lambda}{(1-u)^{\lambda+1}} du \qquad \xi = zu$$

$$\sim -\frac{\sin\pi\lambda}{z^\lambda \pi} \int_0^1 \frac{u^{2\lambda}}{(1-u)^{\lambda+1}} du \quad z^\lambda$$

$$= -\frac{\sin\pi\lambda}{z^\lambda \pi} \frac{\Gamma(2\lambda+1)\Gamma(-\lambda)}{\Gamma(\lambda+1)} z^\lambda$$

$$P_\lambda Q_\lambda' - P_\lambda' Q_\lambda = \frac{\sin\pi\lambda\,\Gamma(2\lambda+1)\,\Gamma(-\lambda)}{z^\lambda \pi\,\Gamma(\lambda+1)} \frac{(\Gamma(\lambda+1))^2}{2\Gamma(2\lambda+2)} 2^{\lambda+1} \left[z^\lambda \frac{d}{dz}\left(\frac{1}{z^{\lambda+1}}\right) - \frac{d}{dz}(z^\lambda)\frac{1}{z^{\lambda+1}}\right]$$

$$= -\frac{\sin\pi\lambda}{\pi} \Gamma(-\lambda)\Gamma(1+\lambda) \frac{\Gamma(2\lambda+1)}{\Gamma(2\lambda+2)} (-2\lambda-1) z^{-2}$$

$$= 1 \cdot \frac{1}{2\lambda+1} (-2\lambda-1) z^{-2} = -z^{-2}$$

$$\Rightarrow \boxed{A = -1}$$

We used $\alpha\Gamma(\alpha) = \Gamma(\alpha+1)$
and $\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin\pi\alpha}$

(iii) $$q_\lambda(z) = \frac{Q_\lambda(z)}{P_\lambda(z)} \Rightarrow q_\lambda'(z) = \frac{P_\lambda(z)Q_\lambda'(z) - P_\lambda'(z)Q_\lambda(z)}{[P_\lambda(z)]^2}$$

$$= \frac{W(z)}{(P_\lambda(z))^2}$$

So $$q_\lambda'(z)[P_\lambda(z)]^2 = W(z) = \frac{A}{z^2-1}$$