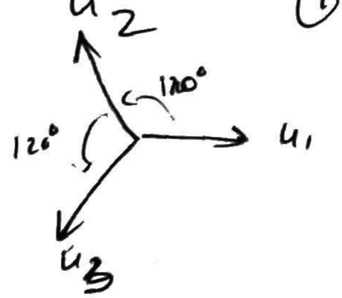
Sol: Probset 3

1. Prob 2 i)



ii) One approach: $x = x_1\phi_1 + x_2\phi_2$

$$\langle u_1, x\rangle = x_1, \quad \langle u_2, x\rangle = -\tfrac{1}{2}x_1 + \tfrac{\sqrt{3}}{2}x_2, \quad \langle u_3, x\rangle = -\tfrac{1}{2}x_1 - \tfrac{\sqrt{3}}{2}x_2$$

$$\sum_k |\langle u_k, x\rangle|^2 = x_1^2 + 2\times\tfrac{1}{4}x_1^2 + 2\times\tfrac{3}{4}x_2^2 = \tfrac{3}{2}(x_1^2+x_2^2)$$ [redacted] $$= \tfrac{3}{2}\|x\|^2$$

$$\Rightarrow \quad \|x\|^2 = \tfrac{2}{3}\sum |\langle u_k, x\rangle|^2$$

iii) $$\sum_k \langle u_k, x\rangle u_k = x_1\phi_1 + \left(-\tfrac{1}{2}x_1 + \tfrac{\sqrt{3}}{2}x_2\right)\left(-\tfrac{1}{2}\phi_1 + \tfrac{\sqrt{3}}{2}\phi_2\right) + \left(-\tfrac{1}{2}x_1 - \tfrac{\sqrt{3}}{2}x_2\right)\left(-\tfrac{1}{2}\phi_1 - \tfrac{\sqrt{3}}{2}\phi_2\right)$$

$$= x_1\phi_1 + \tfrac{1}{2}x_1\phi_1 + \tfrac{3}{2}x_2\phi_2 = \tfrac{3}{2}(x_1\phi_1 + x_2\phi_2) = \tfrac{3}{2}x$$

So $$x = \frac{2}{3}\sum_k \langle u_k, x\rangle u_k$$

Alternative method: $\langle u_i, u_j\rangle = \delta_{ij} - \tfrac{1}{2}(1-\delta_{ij})$

$$\left(\sum_i u_i \otimes u_i^+\right)^2 = \sum_i u_i\otimes u_i^+ \sum_j u_j \otimes u_j^+ = \sum_{ij}\langle u_i, u_j\rangle u_i \otimes u_j^+$$

$$= \sum_i u_i \otimes u_i^+ - \frac{1}{2}\sum_{ij} u_i \otimes u_j^+ + \frac{1}{2}\sum u_i \otimes u_i^+$$

Since $\sum_i u_i = 0$ $\quad \left(\sum_i u_i \otimes u_i^+\right)^2 = \frac{3}{2}\sum_i u_i \otimes u_i^+$

So $\frac{2}{3}\sum_i u_i \otimes u_i^+ = P$ is a projection operator

$\mathrm{Tr}\, P = \frac{2}{3}\sum_i \langle u_i, u_i\rangle = \frac{2}{3}3 = 2$. So it projects to whole two dimension

Therefore, [redacted]

$$x = \frac{2}{3}\sum_i u_i \otimes u_i^\dagger \, x = \frac{2}{3}\sum_i \langle u_i, x\rangle u_i$$

$$\|x\|^2 = \frac{2}{3}\sum_i x^\dagger u_i \otimes u_i^\dagger x = \frac{2}{3}\sum_i |\langle u_i, x\rangle|^2$$

2. Prob 8. M is an invariant mfld of A means for all $x \in M$, $Ax \in M$.

Let $y \in M^\perp \Rightarrow \langle y, x\rangle = 0$ for any $x \in M$

Then for those x $\langle A^\dagger y, x\rangle = \langle y, Ax\rangle = 0$ since $Ax \in M$ as well

Hence $\langle A^\dagger y, x\rangle = 0$ for any $x \in M$.

So $A^\dagger y \in M^\perp$.

$M^\perp$ is an invariant mfld of $A^\dagger$

3. Prob 10. i) $UV(UV)^\dagger = UVV^\dagger U^\dagger$
$= UIU^\dagger = UU^\dagger = I$

So UV is unitary.

ii) $(U+V)(U+V)^\dagger = I$

iff $UU^\dagger + VU^\dagger + UV^\dagger + VV^\dagger = 2I + UV^\dagger + VU^\dagger$

iff $UV^\dagger + VU^\dagger = -I$

Note that $UV^{\dagger}$ are unitary, and therefore normal. Hence $(UV^{\dagger})^{\dagger} = VU^{\dagger}$ shares eigenvectors with $UV^{\dagger}$ and can be diagonalized by the same orthonormal eigenvector system. Since $UV^{\dagger}$ is unitary, if ϕ_j is an eigenvector

$$UV^{\dagger}\phi_j = e^{i\theta_j}\phi_j$$

Using normality of $UV^{\dagger}$, $\quad VU^{\dagger}\phi_j = e^{-i\theta_j}\phi_j$

$$\text{So } (UV^{\dagger} + VU^{\dagger})\phi_j = 2\cos\theta_j\,\phi_j$$

Since $UV^{\dagger} + VU^{\dagger} = -I$

$$2\cos\theta_j = -1$$

$$\cos\theta_j = -\frac{1}{2} \qquad \sin\theta_j = \pm\frac{\sqrt{3}}{2}$$

$$e^{i\theta_j} = e^{2\pi i/3} = \omega \qquad e^{-i\theta_j} = e^{-2\pi i/3} = \omega^2$$

Let us consider the orthonormal eigenvectors of $UV^{\dagger}$ corresponding to the eigenvalue ω. Projection to the subspace spanned by these eigenvectors is P. The other eigenvectors have eigenvalue ω^2. The projection operator to that subspace is $I - P$. Hence $UV^{\dagger} = \omega P + \omega^2(I - P)$.

4 Prob 24. i) $\begin{vmatrix} 1-\lambda & 1 \\ 0 & 1+\varepsilon-\lambda \end{vmatrix} = (\lambda-1)(\lambda-1-\varepsilon)$

Eigenvalue $\lambda=1$, $\begin{pmatrix} 0 & 1 \\ 0 & \varepsilon \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = 0 \Rightarrow$ eigenvector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

" $\lambda = 1+\varepsilon$, $\begin{pmatrix} -\varepsilon & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = 0 \Rightarrow$ eigenvector $\underbrace{\frac{1}{\sqrt{1+\varepsilon^2}}}_{\text{Normalization}}\begin{pmatrix} 1 \\ \varepsilon \end{pmatrix}$

ii) $\cos\theta = \begin{pmatrix} 1 & 0 \end{pmatrix} \frac{1}{\sqrt{1+\varepsilon^2}} \begin{pmatrix} 1 \\ \varepsilon \end{pmatrix} = \frac{1}{\sqrt{1+\varepsilon^2}}$

$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\varepsilon/\sqrt{1+\varepsilon^2}}{1/\sqrt{1+\varepsilon^2}} = \varepsilon$

So $\theta = \tan^{-1}\varepsilon$.

iii) As $\varepsilon \to 0$, $\theta \to 0$, meaning the two eigenvectors merge. This is case where degenerate eigenvalues are associated with a single eigenvector. Note that $A - I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is nilpotent

iv) [redacted] A^+ has the same eigenvalues since $\det\begin{pmatrix} 1-\lambda & 0 \\ 1 & 1+\varepsilon-\lambda \end{pmatrix} = \det\begin{vmatrix} 1-\lambda & 1 \\ 0 & 1+\varepsilon-\lambda \end{vmatrix}$

$\lambda = 1$, $\begin{pmatrix} 0 & 0 \\ 1 & \varepsilon \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = 0 \Rightarrow$ [redacted] eigenvector $\frac{1}{\sqrt{1+\varepsilon^2}}\begin{pmatrix} -\varepsilon \\ 1 \end{pmatrix}$

$\lambda = 1+\varepsilon$, $\begin{pmatrix} -\varepsilon & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = 0 \Rightarrow$ eigenvector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Once more, $\cos\theta = \frac{1}{\sqrt{1+\varepsilon^2}} \Rightarrow \theta = \tan^{-1}\varepsilon$

With same behavior that as $\varepsilon \to 0$ the two eigenvectors become one.

Extra credit!

Prob 27. Straight forward route

i) $\det A = \lambda_1\lambda_2$ [λ's are the eigenvalue]

$\frac{1}{2}[(\mathrm{tr}A)^2 - \mathrm{tr}A^2] = \frac{1}{2}[(\lambda_1+\lambda_2)^2 - (\lambda_1^2+\lambda_2^2)]$

$= \frac{1}{2} 2\lambda_1\lambda_2 = \lambda_1\lambda_2$

ii) $\det A = \lambda_1\lambda_2\lambda_3$

$\frac{1}{6}[(\mathrm{tr}A)^3 - 3(\mathrm{tr}A)(\mathrm{tr}A^2) + 2\,\mathrm{tr}A^3]$

$= \frac{1}{6}\{(\lambda_1+\lambda_2+\lambda_3)^3 - 3(\lambda_1+\lambda_2+\lambda_3)(\lambda_1^2+\lambda_2^2+\lambda_3^2) + 2(\lambda_1^3+\lambda_2^3+\lambda_3^3)\}$

$= \frac{1}{6}[\lambda_1^3+\lambda_2^3+\lambda_3^3 + 3\lambda_1^2\lambda_2 + 3\lambda_2^2\lambda_3 + 3\lambda_3^2\lambda_1 + 3\lambda_2^2\lambda_1 + 3\lambda_3^2\lambda_2 + 3\lambda_1^2\lambda_3 + 6\lambda_1\lambda_2\lambda_3 - 3\lambda_1^3 - 3\lambda_2^3 - 3\lambda_3^3 - 3\lambda_1^2\lambda_2 - 3\lambda_2^2\lambda_3 + 3\lambda_3^2\lambda_1 - 3\lambda_2^2\lambda_1 - 3\lambda_3^2\lambda_1 - 3\lambda_1^2\lambda_3 + 2\lambda_1^3 + 2\lambda_2^3 + 2\lambda_3^3] = \lambda_1\lambda_2\lambda_3$

iii) $\det A = \frac{1}{24}\Big[(\mathrm{tr} A)^4 - 6(\mathrm{tr} A)^2 \mathrm{tr} A^2$

$+ 8 \,\mathrm{tr} A \,\mathrm{tr} A^3 + 3(\mathrm{tr} A^2)^2 - 6\,\mathrm{tr} A^4\Big]$

$$\frac{1}{24}\Big[(\lambda_1+\lambda_2+\lambda_3+\lambda_4)^4 - 6(\lambda_1+\lambda_2+\lambda_3+\lambda_4)^2(\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2)$$
$$+ 8(\lambda_1+\lambda_2+\lambda_3+\lambda_4)(\lambda_1^3+\lambda_2^3+\lambda_3^3+\lambda_4^3) + 3(\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2)^2$$
$$- 6(\lambda_1^4+\lambda_2^4+\lambda_3^4+\lambda_4^4)\Big] = \frac{1}{24}(24\lambda_1\lambda_2\lambda_3\lambda_4)$$
$$= \lambda_1\lambda_2\lambda_3\lambda_4$$

This is a lot of calculation!

Altenalive method!

Consider $\det(I + zA) = \sum_{m=0}^{n} a_m z^m$

For $n \times n$ matrices [redacted] The coefficent of z^n is the determinant, $\det(A)$

Now $\det(I + zA) = e^{\ln \det(I+zA)}$
$= e^{\mathrm{tr} \ln(I+zA)}$

So $\det(I+zA) = e^{\mathrm{tr}\ln(I+zA)}$

$$= e^{\mathrm{tr}\left[zA - \frac{1}{2}z^2A^2 + \frac{1}{3}z^3A^3 - \frac{1}{4}z^4A^4 + \cdots\right]}$$

We need coeffs [redacted] upto z^4.

$$\det(I+zA) = 1 + \left(z\,\mathrm{tr}A - \frac{1}{2}z^2\,\mathrm{tr}A^2 + \frac{1}{3}z^3\,\mathrm{tr}A^3 - \frac{1}{4}z^4\,\mathrm{tr}A^4 + \cdots\right)$$

$$+ \frac{1}{2!}\left(z\,\mathrm{tr}A - \frac{1}{2}z^2\,\mathrm{tr}A^2 + \frac{1}{3}z^3\,\mathrm{tr}A^3 + \cdots\right)^2$$

$$+ \frac{1}{3!}\left(z\,\mathrm{tr}A - \frac{1}{2}z^2\,\mathrm{tr}A^2 + \cdots\right)^3$$

$$+ \frac{1}{4!}\left(z\,\mathrm{tr}A + \cdots\right)^4 + \cdots$$

$$= 1 + z\,\mathrm{tr}A + \frac{z^2}{2}\left[(\mathrm{tr}A)^2 - \mathrm{tr}A^2\right]$$

$$+ \frac{z^3}{6}\left[(\mathrm{tr}A)^3 - 3\,\mathrm{tr}A\,\mathrm{tr}A^2 + 2\,\mathrm{tr}A^3\right]$$

$$+ \frac{z^4}{24}\left[(\mathrm{tr}A)^4 - 6(\mathrm{tr}A)^2\,\mathrm{tr}A^2 + 8\,\mathrm{tr}A\,\mathrm{tr}A^3 + 3(\mathrm{tr}A^2)^2 - 6\,\mathrm{tr}A^4\right]$$

$$+ \cdots$$

The coeffs of z^n is the answer for the n-dimension determinant.

Infact, those coefficent of z^k is $\displaystyle\sum_{\substack{i_1 \cdots i_k \\ \text{are distinct} \\ \text{indices from } \{1,\dots,n\}}} \lambda_{i_1}\cdots\lambda_{i_k}$