Sol: Probset 2 ①

1. For a series $\sum_{k=0}^{\infty} a_k$, consider the ratio $b_k = \left|\frac{a_{k+1}}{a_k}\right|$. If $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \alpha$ in $\mathbb{R}\cup\{+\infty\}$, then condition $\alpha < 1$ decides the radius of convergence for our examples.

a) $b_k = \frac{|z|^2}{(k+1)^2}$. $\lim_{k\to\infty} b_k = 0$ for any $|z|$

So, this series converges absolutely for any z.

The radius of convergence is $+\infty$.

b) $b_k = |z|^2$. $\lim_{k\to\infty} b_k = |z|^2$.

Converges for $|z|<1$. Radius ~~con~~ of convergence is 1.

c) $b_k = (k+1)^2 |z|^2$ $\lim_{k\to\infty} b_k = +\infty$ for $z \neq 0$

$= 0$ for $z = 0$

So, only $z=0$ series converges. The radius of convergence is zero.

[redacted] 2. We will take the power series for $e^{z\cos\theta}$

[redacted] as the definition of the exponential function.

So $\sum_n g_n(\theta)$ [redacted] $= \frac{1}{2\pi}\sum_n \frac{(z\cos\theta)^n}{n!}$ converges

pointwise to $f(\theta) = \frac{1}{2\pi} e^{z\cos\theta}$ for each θ, whatever z is.

This is also an absolutely convergent series.

$$\left| f(\theta) - \sum_{k=0}^{n} g_k(\theta) \right| = \frac{1}{2\pi}\left| \sum_{k=n+1}^{\infty} \frac{(z\cos\theta)^k}{k!} \right|$$

[redacted]

$$\left| \sum_{k=n+1}^{m} \frac{(z\cos\theta)^k}{k!} \right| \leq [redacted] \sum_{k=n+1}^{m} \frac{|z\cos\theta|^k}{k!} \leq \sum_{k=n+1}^{m} \frac{|z|^k}{k!} \qquad m > n+1$$

$\varepsilon_n(z) = \frac{1}{2\pi}\sum_{k=n+1}^{\infty} \frac{|z|^k}{k!}$ exists and is an upper bound of $\left| f(\theta) - \sum_{k=0}^{n} g_k(\theta) \right|$

Since $\varepsilon_n(z) \to 0$ as $n \to \infty$ $\left[\varepsilon_n(z) = \frac{1}{2\pi}\left(e^{|z|} - \sum_{k=0}^{n} \frac{|z|^k}{k!} \right) \right]$, for $\forall \varepsilon > 0$

We ~~can~~ can find an $N(z,\varepsilon)$ s.t. $n > N(z,\varepsilon) \Rightarrow |\varepsilon_n(z)| < \varepsilon$

Note that $N(z,\varepsilon)$ does not depend on θ.

Alternative: If a function φ is given by a power series $\varphi(\omega) = \sum_{n=0}^{\infty} a_n \omega^n$, and the radius of convergence of the series is R, then in any region $|\omega| < r$, with $r < R$, the series converges uniformly in ω. We could have used it here since the radius of conv. of the exponential function series is $+\infty$, and $|z\cos\theta| < |z| < +\infty$.

3. We need show $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{z\cos\theta}\,d\theta = \sum_{k=0}^{\infty} \frac{z^{2k}}{4^k (k!)^2}$

[Notice correction!]

Because of uniform convergence, we can use term by term integration, writing $\int_{-\pi}^{\pi} f(\theta)\,d\theta$ as $\sum_{n=0}^{\infty} \int_{-\pi}^{\pi} g_n(\theta)\,d\theta$. Consider an individual term

$$\int_{-\pi}^{\pi} g_n(\theta)\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(z\cos\theta)^n}{n!}\,d\theta = \frac{z^n}{2\pi\, n!}\int_{-\pi}^{\pi}\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^n d\theta$$

We can expand $(e^{i\theta}+e^{-i\theta})^n$ by binomial expansion. The only term that contributes in ~~the~~ integral is $e^{0\cdot i\theta} = 1$. This happens only if n is even

$(e^{i\theta}+e^{-i\theta})^{2k} = \sum_{r=0}^{2k} \binom{2k}{r} (e^{i\theta})^{2k-r} (e^{-i\theta})^r$. We have to pull out the term $p = k$.

So $\int_{-\pi}^{\pi} g_{2k+1}(\theta)\,d\theta = 0$

$$\int_{-\pi}^{\pi} g_{2k}(\theta)\,d\theta = \frac{z^{2k}}{2\pi(2k)!}\binom{2k}{k}\left(\frac{1}{2}\right)^{2k}\int_{-\pi}^{\pi} d\theta$$

$$= \frac{z^{2k}}{\cancel{(2\pi)}\cancel{(2k)!}}\,\frac{\cancel{(2k)!}}{k!\,k!}\,\frac{1}{2^{2k}}\,\cancel{(2\pi)}$$

$$= \frac{z^{2k}}{2^{2k}(k!)^2}$$

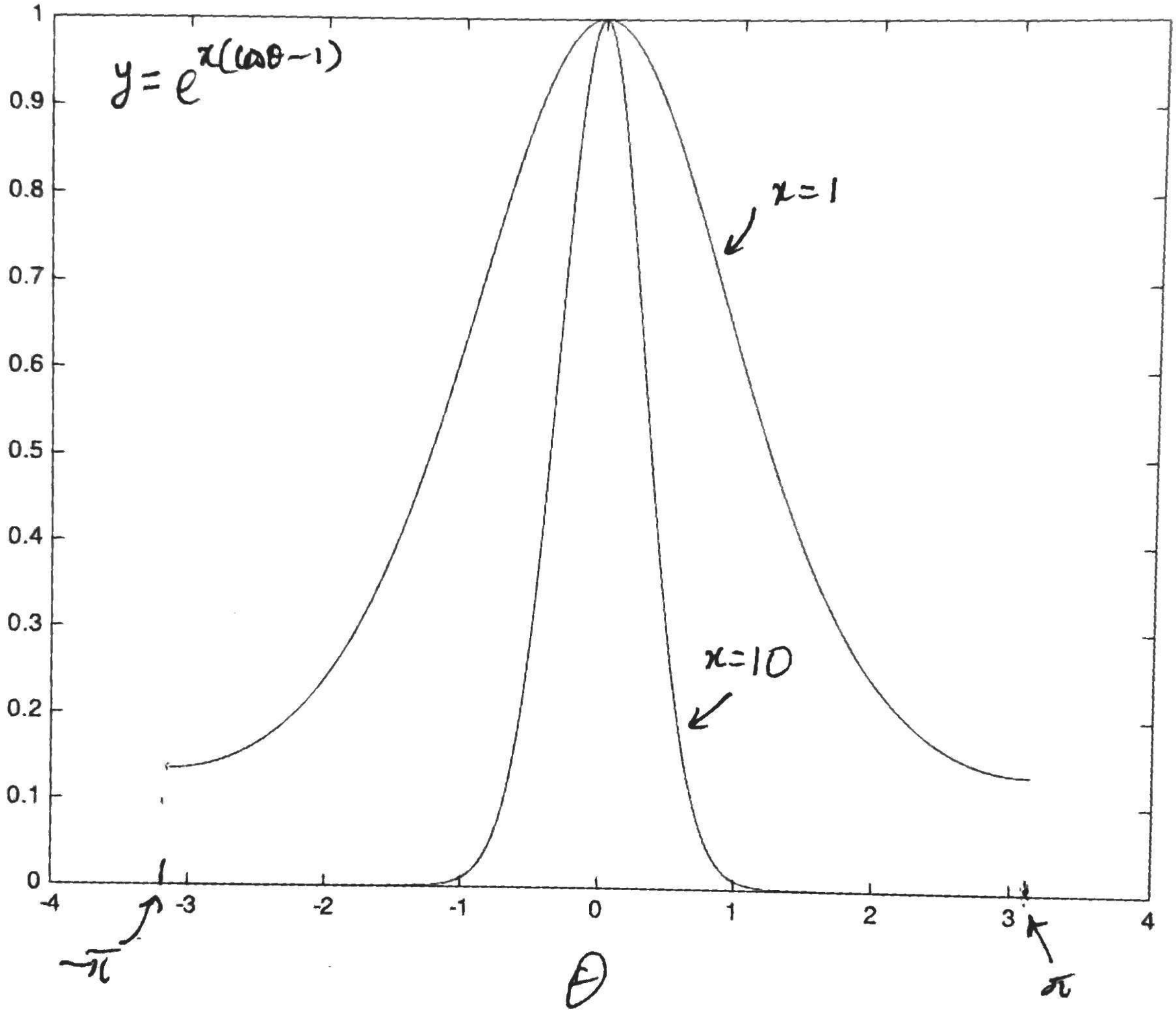
So $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{z\cos\theta}\,d\theta = \sum_{k=0}^{\infty}\frac{z^{2k}}{2^{2k}(k!)^2}$

4. $$I(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x\cos\theta}\,d\theta$$

$$= \frac{e^{x}}{2\pi}\int_{-\pi}^{\pi} e^{x(\cos\theta - 1)}\,d\theta$$

$h(\theta) = e^{x(\cos\theta - 1)}$ is an even function and takes the maximum value at $\theta = 0$ $h(0) = 1$.

[redacted] a) Plots of $y(\theta) = e^{x(\cos\theta - 1)}$ for $x = 1, 10$ below.



The important observation is that, as x goes to larger values, the peak around $\theta = 0$ gets [redacted] narrower.

b) $$I(x) = \frac{e^x}{2\pi}\int_{-\pi}^{\pi} e^{x(\cos\theta - 1)}\, d\theta$$

$$= \frac{e^x}{\pi}\int_0^{\pi} e^{-x(1-\cos\theta)}\, d\theta$$

Call $1-\cos\theta = t$, $\sin\theta\, d\theta = dt$

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-(1-t)^2} = \sqrt{2t-t^2}$$

$$= \sqrt{2t\left(1-\frac{t^2}{2}\right)}$$

At $\theta = 0$ $\cos\theta = 1$ $\Rightarrow t = 0$

At $\theta = \pi$ $\cos\theta = -1$ $\Rightarrow t = 2$

$$I(x) = \frac{e^x}{\pi}\int_0^2 e^{-xt}\frac{1}{\sqrt{2t(1-t/2)}}\, dt$$

$$= \frac{e^x}{\sqrt{2}\,\pi}\int_0^2 e^{-xt}\, t^{-\frac12}\left(1-\frac{t}{2}\right)^{-1/2} dt$$

$$= \frac{e^x}{\sqrt{2}\,\pi}\int_0^2 e^{-xt}\left(t^{-1/2} + \frac14 t^{1/2} + O(t^{3/2})\right) dt$$

$$= \frac{e^x}{\sqrt{2}\,\pi}\left[\frac{1}{x^{\frac12}}\Gamma\left(\frac12\right) + \frac14\frac{1}{x^{3/2}}\Gamma\left(\frac32\right) + O\left(\frac{1}{x^{5/2}}\right)\right]$$ Using Watson's lemma

$$= \frac{e^x}{\sqrt{2}\,\pi\sqrt{x}}\left[\sqrt{\pi} + \frac{1}{x}\,\frac{1}{2}\sqrt{\pi} + O\left(\frac{1}{x^2}\right)\right]$$

$$= \frac{e^x}{\sqrt{2\pi x}}\left(1 + \frac{1}{8x} + O\left(\frac{1}{x^2}\right)\right)$$

So $a = \frac{1}{\sqrt{2\pi}}$, $b = \frac{1}{8\sqrt{2\pi}}$.