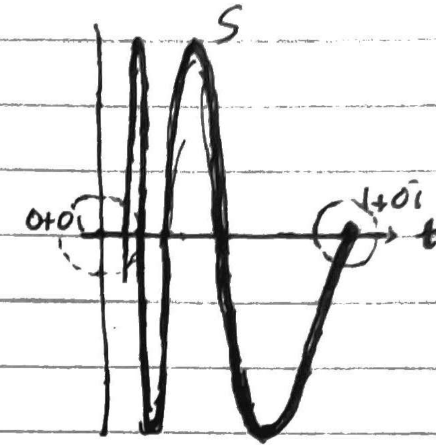
Sol: Probset 1



1. $S = \{ t + i \sin\frac{\pi}{t} \mid t \in (0,1] \}$

a) Consider the point $z_0 = 1 + 0i$

For any $\varepsilon > 0$, $z = 1 + \frac{\varepsilon}{2} + 0i \notin S$ and yet $|z - z_0| = \frac{\varepsilon}{2} < \varepsilon$

So, S is not open.

b) Consider the sequence of points $\{z_n\}$ in S for $t = \frac{1}{n}$, $n = 1, 2, 3, \dots$

$z_n = \frac{1}{n} + i\sin n\pi = \frac{1}{n} + 0i$. Note that $\{z_n\}$ converges to $z = 0 + 0i$. Note that $z \notin S$. If S was closed any convergent sequence inside S would have limit is in S.

So, S is not closed.

2. a) It is a series with positive terms. So the partial sums are monotonically increasing.

If the sequence of partial sums are not bounded then the series tends to $+\infty$.

If the partial sums are bounded, that sequence in convergent, since it is monotonically increasing

In that case the series is convergent.

b) We apply the integral test. Consider the integral $I = \int_3^\infty \frac{dt}{t(\ln t)(\ln(\ln t))^\alpha}$.

Choose $s = \ln(\ln t)$ as a substitution.

$$I = \int_{\ln(\ln 3)}^{\infty} \frac{ds}{s^\alpha}$$

For $\alpha > 1$, $I < +\infty$. So, for $\alpha > 1$,

$$\sum_{n=3}^{\infty} \frac{1}{n(\ln n)(\ln(\ln n))^\alpha}$$ is convergent

When $\alpha \leq 1$, $I = +\infty$. So, for $\alpha \leq 1$,

$$\sum_{n=3}^{\infty} \frac{1}{n(\ln n)(\ln(\ln n))^\alpha}$$ is divergent.

3. We need to prove convergence $\Rightarrow$ Cauchy and Cauchy $\Rightarrow$ Convergence.

Let us assume $\{x_n\} \to x$. Then for any $\varepsilon > 0$, there is an $N \in \mathbb{N}$ s.t. $p > N$ implies $\|x_p - x\| < \frac{\varepsilon}{2}$.

Now for $m, n > N$

$$\|x_m - x_n\| = \|(x_m - x) - (x_n - x)\|$$
$$\leq \|x_m - x\| + \|x_n - x\| \quad \text{, Triangle inequality}$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$
$$= \varepsilon$$

Hence, we can show that $\{x_n\}$ is Cauchy.

For the converse, assume $\{x_n\}$ is Cauchy.
$x_n = (\xi_{1n}, \ldots, \xi_{kn}) \in \mathbb{R}^k$.
For the i-th component, $\{\xi_{in}\}$ is a real sequence.
Since $|\xi_{im} - \xi_{in}| \leq \sqrt{\sum_{j=1}^{k} (\xi_{jm} - \xi_{jn})^2} = \|x_m - x_n\|$

$\{x_n\}$ being Cauchy implies $\{\xi_{in}\}$ is Cauchy for any $i = 1, \ldots, k$. Hence each of those sequences are convergent. Let us say $\{\xi_{in}\} \to \xi_i$, and let x be $(\xi_1, \ldots, \xi_k)$.

For any $\varepsilon > 0$, there is an $N_i \in \mathbb{N}$ s.t. $|\xi_{in_i} - \xi_i| < \frac{\varepsilon}{\sqrt{k}}$, when $n_i > N_i$.

Choose $N = \max(N_1, \ldots, N_k)$.

When $n > N$, $|\xi_{i,n} - \xi_i| < \frac{\varepsilon}{\sqrt{k}}$ for all $i = 1, \dots, k$.

Then $\|x_n - x\| = \sqrt{\sum_{i=1}^{k} (\xi_{i,n} - \xi_i)^2} < \sqrt{\underbrace{\frac{\varepsilon^2}{k} + \dots + \frac{\varepsilon^2}{k}}_{k\text{ terms}}}$

$$= \varepsilon$$

Thus we can show that [redacted] $\{x_n\}$ converges to x.

BTW, if you use Bolzano Weierstrass theorem for $\mathbb{R}^k$, you get partial credit, since several of proofs of Bolzano-Weierstrass depends on completeness of closed bounded sets, which could depend on Cauchy sequences converging in $\mathbb{R}^k$.

Bonus Problem: See the graph on the next page.

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