PROJECT ______

Geometry in Physics

Example 1. Fields in physics: Vector calculus in Electromagnetism, Fluid dynamics, vectors and tensors in elasticity.

Example 2. Spacetime as 4-dim space, Special and general relativity. Geodesics and curvature with metric tensor.

Example 3. Classical mechanics: positions and velocities → 6N dimensional space

Position and momenta (Phase space)

We could deal with $\mathbb{R}^n$ but for some problems we might want to deal with more general "curved" spaces. Hence the motivation to study manifolds.

Example: A space looking like a sphere.

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Before we define a manifold we need a few things from point set topology. We have defined open sets and closed sets in $\mathbb{R}^n$ using a concept of distance. The distance between x, y

$$d(x,y) = \|x-y\| = \sqrt{\langle x-y, x-y\rangle}$$

allowed definition of ε-neighborhoods $N_\varepsilon(x) = \{y \mid d(y,x) < \varepsilon\}$.

An open set U had every point inside it having an ε-neighborhood fully inside the open set

Closed sets were complements of open sets.

These open sets have the property that if U_1 and U_2 are open sets $U_1 \cap U_2$ is open too:

(Proof! For any $p \in U_1$, there is an $\varepsilon_1 > 0$ s.t. $N_{\varepsilon_1}(p) \subset U_1$
If $p \in U_2$, as well, there is an $\varepsilon_2 > 0$ s.t. $N_{\varepsilon_2}(p) \subset U_2$
If $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$ then $N_\varepsilon(p) \subset U_1 \cap U_2$.)

Similarly if $\{U_\alpha \mid \alpha \in A\}$ is a family of indexed open sets, then $\bigcup_{\alpha\in A} U_\alpha$ is open as well.

(Proof! If $p \in \bigcup_{\alpha\in A} U_\alpha$, then $p \in U_\beta$ for some $\beta \in A$. $\exists \varepsilon > 0$, $N_\varepsilon(p) \subset U_\beta \subset \bigcup_{\alpha \in A} U_\alpha$)

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The null set [redacted] ϕ is open since we cannot find any p violating the condition.

$\mathbb{R}^n$ itself is open, since any $p \in \mathbb{R}^n$ has an ε-neighborhood inside $\mathbb{R}^n$.

These properties give rise to an axiomatic definition of a topology.

Def: A topology on a set X is a [redacted] set $\mathcal{T}$ of subsets of X, s.t.

a) If $U_1, U_2 \in \mathcal{T}$, then $U_1 \cap U_2 \in \mathcal{T}$.

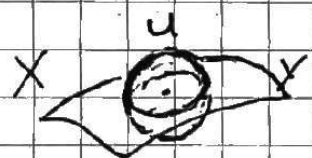
b) If $\{U_\alpha \mid \alpha \in A\} \subset \mathcal{T}$, then $\bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$

c) $\phi \in \mathcal{T}$ and $X \in \mathcal{T}$.

Def: The tuple $(X, \mathcal{T})$ is called a topological space. Sometimes $\mathcal{T}$ is understood.

Def: If $Y \subset X$ and X has a topology $\mathcal{T}$, we

$\mathcal{T}_Y = \{U \cap Y \mid U \in \mathcal{T}\}$ the induced (or relative) topology on Y.



Def: A collection of [redacted] subsets of X, $\mathcal{B}$, is a basis of a topology $\mathcal{T}$ on X if any open set in $\mathcal{T}$ can be formed by unions of sets in $\mathcal{B}$.

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Examples: ε neighborhoods of points for different ε's for a basis of open sets in $\mathbb{R}^n$.

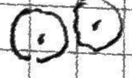
In fact, if we consider points in $\mathbb{R}^n$ with rational coordinates, and consider open balls around such points with radii which are positive rationals, we get a basis for the topology on $\mathbb{R}^n$ we mentioned. This basis is countable, i.e. one can number these sets as

$$B = \{U_i \mid i \in \mathbb{N}\}$$

<u>Def</u>! Spaces with countable bases are called separable.

<u>Def</u>: If X_1, X_2 has topology T_1, T_2, then $X_1 \times X_2$ has the product topology generated by the basis $\{U_1 \times U_2 \mid U_1 \in T_1, U_2 \in T_2\}$

In $\mathbb{R}^n$, in fact, in any metric space, a space with a good distance measure, if $x \neq y$, there are non intersecting open neighborhoods of x, y



<u>Def</u>: If for $\forall x, y \in X$ with $x \neq y$, the topology T has two open sets U and V s.t. $x \in U$, $y \in V$ and $U \cap V = \emptyset$,

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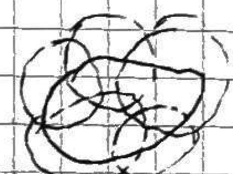
then $(X,\mathcal{T})$ is Hausdorff topological space.

Hausdorff property makes limits unique, for example.

This definition is for completeness. We won't need it soon

Def: If a set $C \subset X$ has the property that any open cover of C, namely $\exists \{U_\alpha \mid \alpha \in A\}$ s.t. $C \subset \bigcup_{\alpha \in A} U_\alpha$, then there is finite subcover $\{U_{\alpha_i} \mid i \in \{1,\ldots,n\}\}$.

That is $C \subset \bigcup_{i \in \{1,\ldots,n\}} U_{\alpha_i}$.



In $\mathbb{R}^n$ closed and bounded sets are the only compact sets.

Def: If X, Y are topological spaces and we have $f: X \to Y$, and any for any open set U in Y, $f^{-1}(U)$ is open in X, then f is a continuous function.

Def: If $f: X \to Y$ is bijection (1-1 and onto) and f and f^{-1} are both continuous, then f is a homeomorphism.

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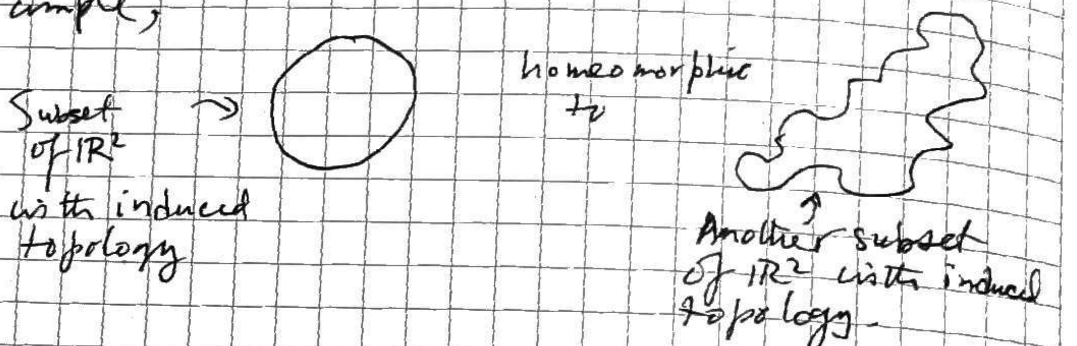
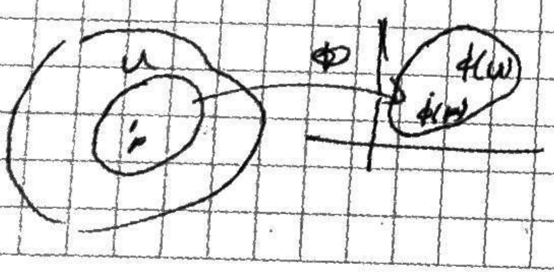
A homeomorphism says that X, Y are essentially equivalent as topological spaces.

For example;



Finally we are ready to define manifolds.

<u>Def</u>: If M is a topological space and $p \in M$, then, a chart at p is a continuous function $\phi : U \to \mathbb{R}^n$, with $p \in U$, and ϕ being a homeomorphism between U and $\phi(U)$



Note that $(\phi_1(p), \ldots, \phi_n(p))$, the components in $\mathbb{R}^n$, are called the coordinates.

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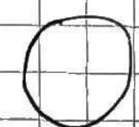
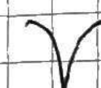
As a result ϕ is often called the coordinate map. The dimension of a chart is n. It is some work show it is unique.

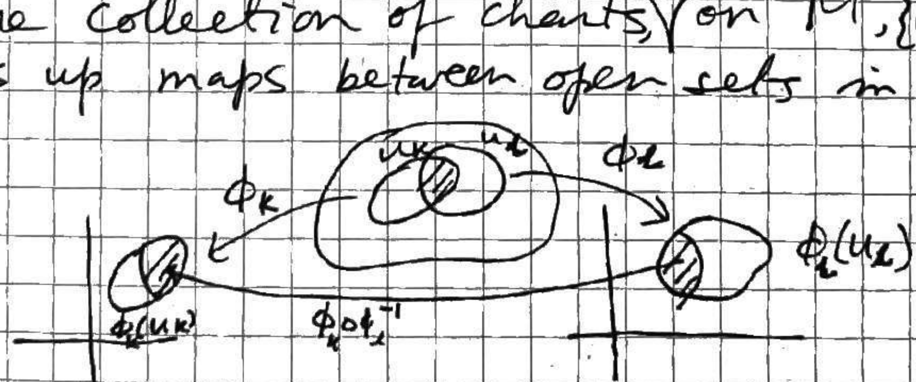
Def: A topological manifold is a separable Hausdorff space with an n-dimensional chart around any point. The dimension of the manifold is n.

In this definition,  is a topological manifold, bu

 is a [redacted] topological manifold of dimension 1 as well.

We need additional structure to, say, differentiate functions on manifolds. We need calculus on manifolds.

The collection of charts, also called an atlas, on M, $\{(U_k, \phi_k) \mid k \in I\}$ sets up maps between open sets in $\mathbb{R}^n$



Note that $\phi_k \circ \phi_\ell^{-1} : \phi_\ell(U_k \cap U_\ell) \longrightarrow \phi_k(U_k \cap U_\ell)$ is a homeomorphism.

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$\phi_k \circ \phi_\ell^{-1}$ is a function ~~from~~ an open subset of $\mathbb{R}^n$ to another open subset of $\mathbb{R}^n$

$\phi_k \circ \phi_\ell^{-1} : (x_1, \dots, x_n) \longmapsto (y_1, \dots, y_n)$

If we demand that these functions [redacted] have all partial derivatives upto kth order (sometimes denoted as C^k) we get a C^k- [redacted] related pair of charts.

[redacted]

Def: An atlas is a C^k atlas if all pairs are C^k related.

Def: A chart in C^k admissible to [redacted] an atlas if it in C^k related to all the charts in the atlas.

Def: A C^k manifold is a topological manifold together with all the admissible charts of some C^k atlas.

For practical purposes, we will be happy with M and an atlas $\{(U_k, \phi_k) \mid k \in I\}$. We won't need the maximal atlas with all other admissible charts.

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