Linear Operators

Now we focus on linear maps from V to V

Def: An operator A (on a vector space V) is a linear map from V to V, $A: V \to V$, with $A(ax+by) = aAx + bAy$ for $x, y \in V$, $a, b \in F$

Note: Infinite dimensional vector spaces might require restriction of the domain of A to a subspace D_A of V.

Note: Nonsingular operator A :

[redacted] If $\ker(A) = \{\theta\}$, and $y = Ax$, for $x, y \in V$, then $A^{-1}: V \Rightarrow V$, is defined by $A^{-1}y = x$.

One can show that this a well defined linear map.

[redacted] One can also show $(AB)^{-1} = B^{-1}A^{-1}$ when A, B are both nonsingular.

Def: Adjoint operator: $A: V \to V$ naturally [redacted] gives rise to an operator $A^*: V^* \to V^*$ as follows. For a linear functional $\lambda \in V^*$

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$\Lambda : V \to \mathbb{C}$

$A^*\Lambda : V \to \mathbb{C}$ defined via $(A^*\Lambda)(x) = \Lambda(Ax)$

If we have a unitary vector space and we focus on linear functionals of the form $\langle u, \bullet \rangle$, the we define A^+ via

$$\langle u, Ax \rangle = \langle A^+ u, x \rangle .$$

Some easy consequences: If $x \in Ker(A)$

$$\langle u, Ax \rangle = 0 \iff \langle A^+ u, x \rangle = 0$$

$$\Rightarrow Ker(A) = (im(A^+))^{\perp}$$

Same way $Ker(A^+) = (im(A^+))^{\perp}$

Def: An operator is self-adjoint or Hermitian if $A = A^+$.

We will come back to properties of self-adjoint operators soon.

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Change of Basis

Let $\{x_1, x_2, \dots\}$ is a basis and $\{y_1, y_2, \dots\}$ is ~~another~~ Define a linear operator S so that

$$S x_k = y_k = \sum_j S_{jk} x_j$$

Let a [redacted] vector $x = \sum_k \xi_k x_k = \sum_l \eta_l y_l$. How [redacted] are the component ξ_k's and η_l's related?

$$\sum_l \eta_l y_l = \sum_l \eta_l \sum_k S_{kl} x_k = \sum_k \Big(\sum_l S_{kl} \eta_l\Big) x_k$$

So $\xi_k = \sum_l S_{kl} \eta_l$

If we use the nonsingularity of S, $x_k = S^{-1} y_k = \sum_j \bar{S}_{jk} y_j$

Same argument $\Rightarrow$ $\eta_l = \sum_k \bar{S}_{lk} \xi_k$

matrix elements

$$A x_k = \sum_j A_{jk} x_j$$

How does it look in in $\{y_l\}$ basis?

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$$Ay_k = A\sum_j S_{jk} x_j = \sum_j S_{jk} A x_j = \sum_{jl} S_{jk} A_{lj} x_l$$

$$= \sum_{jlm} S_{jk} A_{lj} \bar{S}_{ml} y_m = \sum_m \underbrace{\left(\bar{S}_{ml} A_{lj} S_{jk}\right)} y_m$$

Same matrix elements as for $S^{-1}AS$.

To be more careful, some authors prefer indicate matrices as $M(A, (x_1, \ldots, x_n))$ or $M(A, (y_1, \ldots, y_n))$

operator, ordered basis; same operator, another ordered basis

Change of basis, in which scalar products are preserved in an unitary vectors space V, have the property $\langle Sx, Sy \rangle = \langle x, y \rangle$

Using an orthonormal basis, one could show $S^\dagger S = I$. This are unitary transformations (for real it is orthogonal).

Example: $x_1, x_2 \leftrightarrow \hat{i}, \hat{j}$

$y_1 = x_1\cos\theta + x_2\sin\theta$

$y_2 = -x_1\sin\theta + x_2\cos\theta$

$$[S_{jk}] = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = R(\theta)$$

Invariant (linear) manifolds (or linear subspaces)

Def: A linear manifold M in V, an n-dimensional vector space, is an invariant manifold if, for any $x \in M$, $Ax \in M$.

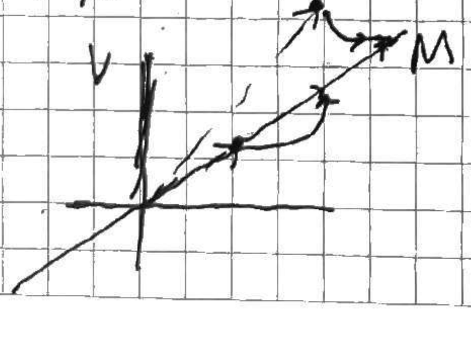
If M has dimension m, we have a basis $x_1, \dots, x_m$. This basis can be extended to $x_1, \dots, x_m, x_{m+1}, \dots, x_n$ to form a basis of V.

$V = M(x_1, \dots, x_m) + M^*(x_{m+1}, \dots, x_n) = M \oplus M^*$

The matrix form of A looks like $\begin{pmatrix} A_M & B \\ 0 & A_{M^*} \end{pmatrix}$

Vectors in $M \rightarrow \begin{pmatrix} - \\ - \\ - \\ - \\ 0 \\ 0 \\ 0 \end{pmatrix}$ } only first m components could be nonzero.

A_M is related to the restriction of A to M.



Examples of invariant mfld.

$\ker(A^k)$, $\mathrm{im}(A^k)$ $\quad k = 0, 1, 2, \dots$

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$$A^k(Ax) = A(A^k x)$$

If $A^k x = 0$ then $A^k(Ax) = 0$. In other words $x \in \ker(A^k) \Rightarrow Ax \in \ker(A^k)$

If $y = A^k x$ then $Ay = A^k(Ax)$. In other words $y \in im(A^k) \Rightarrow Ay \in im(A^k)$

Moreover $\ker(A^0) = \{\theta\} \subseteq \ker(A) \subseteq \ker(A^2) \subseteq \cdots$

and $im(A^0) = V \supseteq im(A) \supseteq im(A^2) \supseteq \cdots$

Since V is finite dimensional, there must be some smallest p, s.t. $\ker(A^p) = \ker(A^{p+1})$. Otherwise the dim keeps on increasing.

Since $\dim(\ker(A^k)) + \dim(im(A^k)) = n$, if $\dim(\ker(A^p)) = \dim(\ker(A^{p+1}))$
$\dim(~~ker~~(A^p)) = \dim(im(A^{p+1})) \Rightarrow im(A^p) = im(A^{p+1})$.

How $im(A^{k+1})$ be a proper subset of $im(A^k)$? There has to be some nonzero vector $y \in im(A^k)$ so that $Ay = 0$. $y = A^k x \neq \theta$ but $Ay = A^{k+1}x = \theta$. When $k \geq p$, this is impossible.

Now note that $\ker(A^p) \cap im(A^p) = \{\theta\}$
[$y \in im(A^p) \Rightarrow y = A^p x$, $x \in V$, $y \in \ker(A^p) \Rightarrow \theta = A^p A^p x = A^{2p} x$.
So $x \in \ker(A^{2p}) = \ker(A^p)$. Hence $y = A^p x = \theta$]

So $V = im(A^p) \oplus \ker(A^p)$

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Since each is an invariant manifold matrix of A, written in a basis combining bases of im A^p and ker A^p, looks like

$$\begin{array}{c} \begin{pmatrix} \bar{A} & 0 \\ 0 & N \end{pmatrix} \begin{array}{l} \} \text{ im } A^p \text{ basis} \\ \} \text{ ker } A^p \text{ basis} \end{array} \\ \begin{array}{cc} \underbrace{\quad}_{\text{im } A^p \text{ basis}} & \underbrace{\quad}_{\text{ker } A^p \text{ basis}} \end{array} \end{array}$$

Not that $N^p = 0$. Nilpotent matrix (operator).

Example of Nilpotent matrix : $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \leftarrow \sigma^+ = \frac{\sigma_x + i\sigma_y}{2}$

$N^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. $N \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $N \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

We note that $\dim(\ker(N)) < \dim(\ker(N^2))$

If $\dim(\ker(N)) < \dim(\ker(N^2)) < \cdots < \dim(\ker(N^p))$

We can setup a basis in which N becomes upper triangular

$$N = \begin{pmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & 0 & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Projection Operator in a Unitary Vectors space

$V = M \oplus M^\perp$

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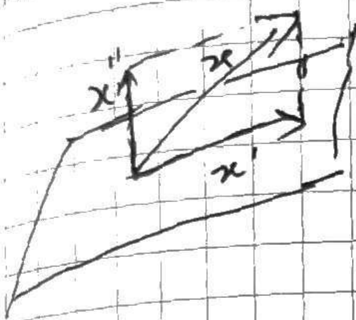

$$x = x' + x''$$

$$x' \in M, \quad x'' \in M^{\perp}$$

$x \mapsto x'$ is a (orthogonal projection map)

$$P_M x = x'$$

Note that $P(Px) = Px$ so $P^2 = P$

Also P is Hermitian. Say $x = x' + x''$ with $x' \in M$, $x'' \in M^{\perp}$

and $y = y' + y''$ with $y' \in M$, $y'' \in M^{\perp}$

$$\langle y, Px \rangle = \langle y' + y'', x' \rangle = \langle y', x' \rangle + \underbrace{\langle y'', x' \rangle}_{\text{zero}} = \langle y', x' \rangle$$

$$\langle Py, x \rangle = \langle y', x' + x'' \rangle = \langle y', x' \rangle + \langle y', x'' \rangle = \langle y', x' \rangle$$

So $\langle y, Px \rangle = \langle Py, x \rangle$.

Note that $I - P_M = P_{M^{\perp}}$ and it is also a projection.

$$I = P_M + P_{M^{\perp}}$$ is a resolution of identity.

We have set up the machinery needed to understand, eigenvalues, eigenvectors, diagonalization and properties of normal operators.

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Eigenvectors and Eigenvalues

Def.: If $Ax = \lambda x$ for some $x \in V$, $x \neq \theta$, then x is an eigenvector and λ is the corresponding eigenvalue.

We have not discussed determinants, but you know that $p_A(\lambda) = \det(A - \lambda I) = 0$ gives an n-th order polynomial $p_A(\lambda)$ in n-dim vector spaces and the zeros of the polynomial are the eigenvalues.

If $\{\lambda_1, \dots, \lambda_m\}$ are distinct eigenvalues of A, that is called the spectrum. The corresponding eigenvectors $\{x_1, \dots, x_m\}$ can be shown to be linearly independent.

In general $p_A(\lambda) = (-1)(\lambda - \lambda_1)^{p_1} \cdots (\lambda - \lambda_m)^{p_m}$ with $p_1 + \dots + p_m = n$

The positive integers $p_1, \dots, p_m$ are the algebraic multiplicities of the eigenvalues $\lambda_1, \dots, \lambda_m$.

By the way, if we change the basis, $p_A(\lambda)$ does not get affected.

$$\det[S^{-1}AS - \lambda I) = \det(S^{-1}(A - \lambda I)S)$$
$$= \det(S)\det(S^{-1})\det(A - \lambda I).$$

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Def. The operator A is diagonalizable if its eigenvectors form a basis.

Say $Ax_k = \lambda_k x_k$ and $x_1, \ldots, x_n$ is a basis. Then the matrix representation of A looks like

$$\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$$

Are operators diagonalizable in general? Not quite! We could have problems when we have $p_k > 1$.

Example: $A = \begin{pmatrix} \mu & 1 \\ 0 & \mu \end{pmatrix}$, $V = \mathbb{R}^2$

$p_A(\lambda) = (\lambda - \mu)^2$. Has only one eigenvector: $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Note that $\ker((A - \mu I)^2) = \mathbb{R}^2$ $\quad \ker(A - \mu I) = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \middle| x \in \mathbb{R} \right\}$

Thm: If V is a finite dimensional vector space and the operator A has n distinct eigenvalues $\lambda_1, \ldots, \lambda_m$ with multiplicities $p_1, \ldots, p_m$, then $V = \overline{M}_1 \oplus \overline{M}_2 \oplus \cdots \oplus \overline{M}_m$ with $\dim(\overline{M}_k) = p_k$, where is an invariant

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is an invariant manifold of $(A-\lambda_k I)$ ~~and the~~

restriction of A to $\overline{M}_k$, A_k, is $\lambda_k \mathbb{1}_{p_k} + N_k$, where N_k is a nilpotent matrix.

Proof idea! Look at the kernel of $(A-\lambda_k I)^j$, $j=1,2,\ldots$

Using this we can write A's matrix as

$$\begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_m \end{pmatrix} = \begin{pmatrix} \boxed{\lambda_1 I_{p_1}} & & & \\ & \boxed{\lambda_2 I_{p_2}} & & \\ & & \ddots & \\ & & & \boxed{\lambda_k I_{p_k}} \end{pmatrix} + \begin{pmatrix} \boxed{N_1} & & & \\ & \boxed{N_2} & & \\ & & \ddots & \\ & & & \boxed{N_k} \end{pmatrix}$$

$$= \underset{\text{(diagonal)}}{D} + \underset{\text{(nilpotent)}}{N}$$

D, N commute with each other, meaning $DN = ND$

This result is the basis of the Jordan normal form.

$$A \to \begin{pmatrix} \begin{matrix} \lambda_1 & 1 & 0 \\ & \ddots & 1 \\ 0 & & \lambda_1 \end{matrix} & \\ & \boxed{\begin{matrix} \lambda_1 & 1 & \\ & \ddots & \\ & & \lambda_1 \end{matrix}} \end{pmatrix}$$

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When do we not have these [redacted] nilpotent parts ?

For unitary Vector spaces, it turns out that [redacted] (an important class of operators) [redacted], normal operators, do not have nilpotent parts.

Def : A is a normal operator if $A^+A = AA^+$

Example: a) Hermitian A, since $A = A^+$ $\quad A^+A = A^2 = AA^+$

b) Unitary A, since A^+[redacted]$A = I$, $A^+ = A^{-1}$

If left inverse is the same as the right inverse

A[redacted]$A^+ = AA^{-1} = I = A^+A$

Some properties:

Obs 1 Unitary vector space V; $B : V \to V$ is an operator then $\ker(B) = \ker(B^+B)$

Obs 2 Normal operator A.

(i) $\|Ax\| = \|A^+x\|$

(ii) If A is normal then $A - \lambda I$ is normal

(iii) If A has an eigenvalue λ, the A^+ has the eigenvalue λ^*. with the same eigenvector $Ax = \lambda x \Rightarrow A^+x = \lambda^* x$

(iv) If $Ax_1 = \lambda_1 x_1$ and $Ax_2 = \lambda_2 x_2$, and $\lambda_1 \neq \lambda_2$ then $\langle x_1, x_2 \rangle = 0$

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(v) $\ker(A^k) = \ker(A^{k-1})$ for $k \geq 2$ (Hint: $B = A^{k-1}$, Show $B^2 = 0$ on $\ker A^k$; $(B^+)B^2 = (B^+B)^+(B^+B)$, so $B^+B = 0$ on $\ker A^k$)

(vi) $\ker(A^k) = \ker(A)$

These lead to the conclusion that nilpotent normal operators are null operators.

Fundamental Theorem on Normal operators

If a linear operator A on an n dimensional unitary vector space is normal, then A has a complete orthonormal system of eigenvectors

Proof idea: $V = \bar{M}_1 \oplus \dots \oplus \bar{M}_m$

Restriction of A in $\bar{M}_k$, A_k, is still normal. $A_k - \lambda_k I_{p_k}$ is normal and nilpotent, hence, zero. Now show $\bar{M}_k$ and $\bar{M}_l$ for $k \neq l$ are orthogonal subspaces. Set up orthonormal bases inside each $\bar{M}_k$.

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