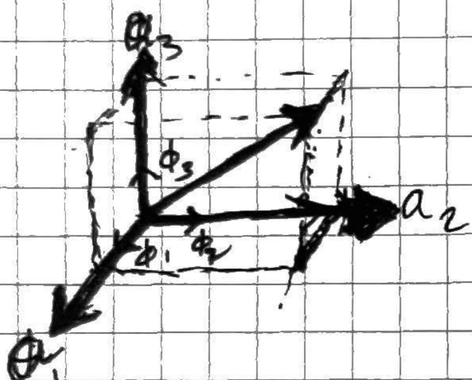
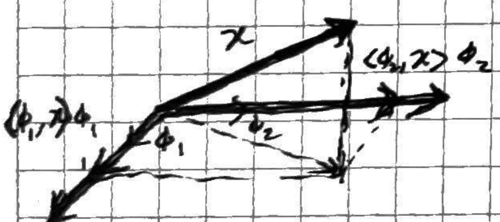
Notebook No.
Continued From Page

Thus $x = \langle \phi_1, x \rangle \phi_1 + \langle \phi_2, x \rangle \phi_2 + \cdots + \langle \phi_m, x \rangle \phi_m$



$\langle \phi_k, x \rangle \phi_k$ is the projection of the vector x in the direction of ϕ_1

What if ~~an~~ we have orthonormal system, [redacted] not necessarily a basis, and consider the projection of a vector x on the subspace spanned by the system?



Bessel's Inequality: $\sum_{k=1}^{m} \langle \phi_k, x \rangle^2 \leq \|x\|^2$

Proof: $\| x - \sum_k \langle \phi_k, x \rangle \phi_k \|^2 \geq 0$

In other words,

[redacted] $\langle x - \sum_k \langle \phi_k, x \rangle \phi_k, x - \sum_l \langle \phi_l, x \rangle \phi_l \rangle \geq 0$

$\Rightarrow \langle x, x \rangle - \sum_l \langle \phi_l, x \rangle \langle x, \phi_l \rangle - \sum_k \langle \phi_k, x \rangle \langle x, \phi_k \rangle$

$\quad + \sum_{k,l} \langle \phi_k, x \rangle^* \langle \phi_l, x \rangle \underbrace{\langle \phi_k, \phi_l \rangle}_{\text{This is } \delta_{kl}} \geq 0$

$\Rightarrow \|x\|^2 - \sum_l \langle \phi_l, x \rangle \langle \phi_l, x \rangle^* - \sum_k \langle \phi_k, x \rangle \langle \phi_k, x \rangle^* + \sum_k \langle \phi_k, x \rangle^* \langle \phi_k, x \rangle \geq 0$

$\Rightarrow \|x\|^2 - \sum_k |\langle \phi_k, x \rangle|^2 \geq 0$ [redacted]

[redacted] $\Rightarrow \|x\|^2 \geq \sum_k |\langle \phi_k, x \rangle|^2$. QED

Continued on Page

Note that Bessel's inequality is valid even in infinite dimensional spaces.

One easy consequence of Bessel's inequality is the Cauchy-Schwartz or the Cauchy-Bunyakovsky-Schwarz inequality. Historical note: [redacted] Cauchy's proof applied to finite dimensional vector spaces where the other two expanded it to the infinite dimensional function spaces.

Cauchy-Schwartz inequality: $|\langle x, y\rangle| \leq \|x\| \|y\|$

Proof: If x [redacted] $= \theta$ Then $\langle x, y\rangle = 0$

$(\langle x+y, y\rangle = \langle y, y\rangle \Rightarrow \langle x, y\rangle + \langle y, y\rangle = \langle y, y\rangle \Rightarrow \langle x, y\rangle = 0)$

If $x \neq \theta$, $\|x\| > 0$. Then define $\phi_1 = \frac{1}{\|x\|} x$

[redacted] and apply Bessel's inequality to y

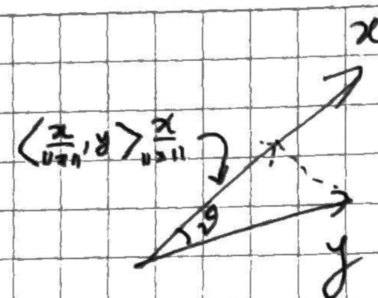
$$|\langle \phi_1, y\rangle|^2 \leq \|y\|^2$$

$$\Rightarrow \left|\left\langle \frac{x}{\|x\|}, y\right\rangle\right|^2 \leq \|y\|^2 \Rightarrow \frac{|\langle x, y\rangle|^2}{\|x\|^2} \leq \|y\|^2$$

$\Rightarrow |\langle x, y\rangle|^2 \leq \|x\|^2 \|y\|^2 \Rightarrow |\langle x, y\rangle| \leq \|x\| \|y\|$ by taking sqrt

QED.

Notebook No.
Continued From Page



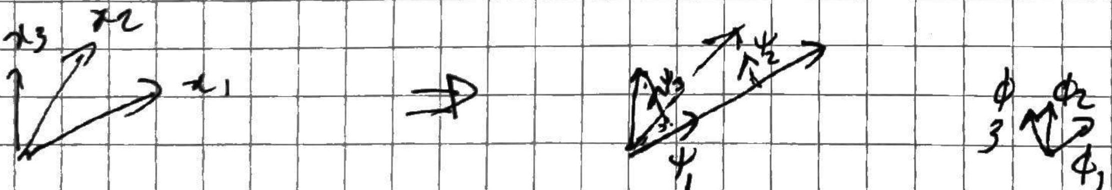
Cauchy-Schwarz allows us to define an angle ϑ between nonzero vectors over reals via

$$\langle x, y\rangle = \|x\| \|y\| \cos\vartheta$$

If you have a set of linearly independent vectors, one can construct an orthonormal system out of it.

Gram-Schmidt orthogonalization process

$\{x_1, \ldots, x_m\}$ linearly independent
(implies they are nonzero vectors)



Idea. Pick the part of x_k orthogonal to the subspace spanned by $\{x_1, \ldots, x_{k-1}\}$.

Precisely: $\phi_1 = \frac{x_1}{\|x_1\|}$

$$\underbrace{\psi_k = x_k - \sum_{p=1}^{k-1} \langle \phi_p, x_k\rangle \phi_p}_{\text{"Residual"}} \xrightarrow{\text{normalize}} \phi_k = \frac{\psi_k}{\|\psi_k\|}$$

Continued on Page

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(We know that $\|\psi_k\| \neq 0$. Otherwise $\{\phi_1, \dots, \psi_k\}$ [redacted] would be linearly dependent).

The linear manifolds spanned by $\{\phi_1, \dots, \phi_k\}$ is the same as the one spanned by $\{x_1, \dots, x_k\}$

$$\mathcal{M}(\phi_1, \dots, \phi_k) = \mathcal{M}(x_1, \dots, x_k)$$

__Def__ The orthonormal system $\phi_1, \phi_2, \dots$ is complete if [redacted] only vector orthonormal to all the vectors in the system is the zero vector; namely, $\langle \phi_k, x \rangle = 0, \forall k$

$$\Rightarrow \quad x = \theta$$

__Thm__ The orthonormal system $\phi_1, \dots, \phi_m$ in an n-dim vector space V is complete

iff
i) $m = n$
ii) $\{\phi_1, \dots, \phi_m\}$ is a basis
(iii) $\|x\|^2 = \sum_k |\langle \phi_k, x \rangle|^2$ [redacted]
(iv) For every pair $x, y \in V$

$$\langle x, y \rangle = \sum_{k=1}^{m} \langle x, \phi_k \rangle \langle \phi_k, y \rangle$$

Continued on Page

Proof: Assume $\phi_1, \phi_2, \dots, \phi_m$ is a complete orthonormal system

Consider $z = x - \sum_{k=1}^{m} \langle \phi_k, x \rangle \phi_k$

$\langle \phi_k, z \rangle = 0$ for $k = 1, \dots, m$

Hence $z = 0 \Rightarrow x = \sum_{k=1}^{m} \langle \phi_k, x \rangle \phi_k$. Very Important!

So any x can be expanded [redacted] as a linear combination of ϕ_k. Since $\{\phi_1, \dots, \phi_m\}$ are linearly independent. Hence $\{\phi_1, \dots, \phi_m\}$ is a basis

So, Completeness $\Rightarrow$ (ii)

If $\{\phi_1, \dots, \phi_m\}$ is a basis and V is n dimensional then $m = n$.

(ii) $\Rightarrow$ (i)

[redacted]

If we have an orthonormal system with n vectors, $\{\phi_1, \dots, \phi_n\}$ in an n dimensional space, we can start with a basis $\{x_1, \dots, x_n\}$ and use the replacement trick to show $\{\phi_1, \dots, \phi_n\}$ is a basis.

So, for any x, $x = a_1 \phi_1 + \dots + a_n \phi_n$. Then $\langle \phi_k, x \rangle = 0$ for all k $\Rightarrow a_k = 0 \Rightarrow x = 0$

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So (i) $\Rightarrow$ completeness

$$\begin{array}{ccc} & \text{completeness} & \\ \nearrow & & \searrow \\ (i) & \Leftarrow & (ii) \end{array}$$

So completeness, (i) & (ii) are equivalent

Once more assume completeness of the orthonormal system $\{\phi_1, \ldots, \phi_m\}$

Consider $x, y \in V$ [redacted], [redacted] $x = \sum_{k=1}^{m} \langle \phi_k, x \rangle \phi_k$, $y = \sum_{k=1}^{m} \langle \phi_k, y \rangle \phi_k$ } using ~~our~~ previous observation.

$$\langle x, y \rangle = \left\langle \sum_{k=1}^{m} \langle \phi_k, x \rangle \phi_k , \sum_{l=1}^{m} \langle \phi_l, y \rangle \phi_l \right\rangle$$

$$= \sum_{k=1}^{m} \langle \phi_k, x \rangle^* \langle \phi_l, y \rangle \underbrace{\langle \phi_k, \phi_l \rangle}_{\delta_{kl}}$$

$$= \sum_{k=1}^{m} \langle \phi_k, x \rangle^* \langle \phi_k, y \rangle$$

$$= \sum_{k=1}^{m} \langle x, \phi_k \rangle \langle \phi_k, y \rangle$$

So completeness $\Rightarrow$ (iv)

If we [redacted] have [redacted] $\langle x, y \rangle = \sum_{k=1}^{m} \langle x, \phi_k \rangle \langle \phi_k, y \rangle$

then, setting, $y = x$ $\quad \|x\|^2 = \sum_{k=1}^{m} |\langle \phi_k, x \rangle|^2$

So (iv) $\Rightarrow$ (iii)

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Now assume $\|x\|^2 = \sum_{k=1}^{m} |\langle \phi_k, x \rangle|^2$ for any $x \in V$.

If $\langle \phi_k, x \rangle = 0$ for all k.

$\|x\|^2 = 0 \implies x = 0$

Hence $\{\phi_1, \dots, \phi_m\}$ is a complete orthonormal system.

[redacted] So, (iii) $\implies$ Completeness!

$$\begin{tikzcd}
(iv) \arrow[dd, Rightarrow] & & (ii) \arrow[dd, Rightarrow] \\
& \text{completeness} \arrow[ul, Rightarrow] \arrow[ur, Rightarrow] & \\
(iii) \arrow[ur, Rightarrow] & & (i) \arrow[ul, Rightarrow]
\end{tikzcd}$$

[redacted] Thus, all the statements are equivalent.

Note that if we have a finite dimensional vector space, we have [redacted] basis with n vectors. We can then form an orthonormal basis with n vectors by the Gram-Schmidt orthogonalization process.

Continued on Page

Def. Direct Sum of two vector spaces U and V over the same field F could be constructed as follows.

The direct sum $U \oplus V$ [redacted] consists of ordered pairs (u, v) with $u \in U$, $v \in V$, with addition defined by $(u_1, v_1) + (u_2, v_2) = (u_1 + u_2, v_1 + v_2)$ and scalar multiple defined by $a \cdot (u, v) = (a \cdot u, a \cdot v)$ for $a \in F$. One can show $U \oplus V$ is a vector space.

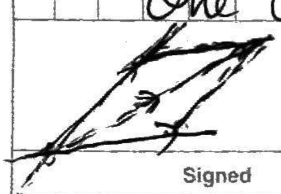
Def: Sum of vector subspaces M_1, M_2 of vector space V is defined as [redacted]

$$M_1 + M_2 = \{ x_1 + x_2 \mid x_1 \in M_1, x_2 \in M_2 \}.$$

One can easily show $M_1 + M_2$ is a vector space.

Under some conditions $M_1 + M_2$ is 'equivalent' to $M_1 \oplus M_2$, but [redacted] these vector spaces are not equivalent in general. The equivalence has to be defined carefully. (Later!)

One condition of equality is that $M_1 \cap M_2 = \{\theta\}$



Def: For vector space V and a subspace M, the orthogonal complement $M^\perp$ is defined as $\{x \in V \mid \langle x, y \rangle = 0 \text{ for all } y \in M\}$.

(One can show $M^\perp$ is a vector subspace. [redacted]

Obs 1 $\quad M \cap M^\perp = \{\theta\}$

If $x \in M$ and $x \in M^\perp$ $\quad \langle x, x \rangle = 0$

$\Rightarrow$ [redacted] $\|x\|^2 = 0 \Rightarrow x = \theta.$

(Projection)

[redacted] Thm If V is finite dimensional

$$M + M^\perp = V$$

If V is finite dimensional then M is finite dimensional. Otherwise, [redacted] choose [redacted] maximal set I of linearly independent vectors in M. I would [redacted] be linearly independent in V. [redacted] Then it cannot have more than $\dim(V)$ elements. So [redacted] $\dim(M) \le \dim(V)$

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Now, say $\dim(M) = m$ and $\{\phi_1, \ldots, \phi_m\}$ is an orthonormal basis of M.

For any $x \in V$, call $\sum_{k=1}^{m} \langle \phi_k, x \rangle \phi_k = x'$ the projection of x to M. Note that

[redacted] $\langle x - x', \phi_k \rangle = 0$ for all k.

Since any $y \in M$ is a linear combination of ϕ_k's

$$\langle x - x', y \rangle = 0 \text{ for all } y \in M$$

So $x - x' = x'' \in M^{\perp}$

Hence any [redacted] $x \in V$ can be written as $x = x' + x''$ with $x' \in M$ and $x'' \in M^{\perp}$

So $M + M^{\perp} = V$ QED

<u>Obs</u> Since $M \cap M^{\perp} = \{0\}$

$M + M^{\perp}$ is 'equivalent' to $M \oplus M^{\perp}$

Continued on Page

We will discuss Product Spaces when we go to tensors later.

<u>Sequences of vectors</u> $\{x_n\} \equiv (x_1; x_2; \dots)$ [redacted] with $x_k \in V$, generalizes our earlier definitions on $\mathbb{R}$ & $\mathbb{C}$.

For a normed vector space, we could define the convergence [redacted] of sequences, or Cauchy sequences just by realizing $z \to |z|$ is one particular norm.

Convergence of $\{x_n\}$ to x for vectors means for any $\varepsilon > 0$, there is an $N \in \mathbb{N}$ s.t. $n > N$ implies $\|x_n - x\| < \varepsilon$

The sequence $\{x_n\}$ being Cauchy means for any $\varepsilon > 0$, there is an $N \in \mathbb{N}$ s.t. $m, n > N$ implies $\|x_m - x_n\| < \varepsilon$.

For $\mathbb{R}^p$ or $\mathbb{C}^p$, these properties are equivalent to same properties for their components. Infinite dimensional spaces open up more interesting possibilities.