Vector Spaces: Abstract Formulation:

A (Linear) Vector Space over a field F (for us $\mathbb{R}$ and $\mathbb{C}$) have following operations

Addition: $V \times V \to V$
$(x, y) \mapsto x+y$

Scalar multiplication: $F \times V \to V$
$(a, x) \mapsto a\cdot x$

under this it is an abelian group
$x+y = y+x$ $\quad x+(y+z) = (x+y)+z$ etc.
There is a θ, $x+\theta = x$
There is a $-x$, s.t. $x+(-x) = \theta$

Scalar multiplication has associativity: $a(bx) = (ab)x$

Very importantly there is the distributive laws
$a\cdot(x+y) = a\cdot x + a\cdot y$ and $(a+b)\cdot x = a\cdot x + b\cdot x$
Unitary Law: $1\cdot x = x$

<u>Observations</u>: 0 and 1 of the field F ($\mathbb{R}$ or $\mathbb{C}$ for us) have the properties $(-1)\cdot x = -x$ and $0\cdot x = \theta$.

Note:
$x = 1\cdot x = (1+0)\cdot x = 1\cdot x + 0\cdot x = x + 0\cdot x$
Add $-x$ on both sides
$\theta = 0\cdot x$. QED

Then
$\theta = 0\cdot x = (1+(-1))\cdot x = 1\cdot x + (-1)\cdot x = x + (-1)x$

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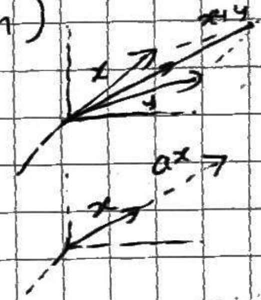
Adding $(-x)$ on both sids

$(-x) = (-1)\cdot x$. QED

Examples i) $\mathbb{R}^n$ over $\mathbb{R}$

$x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$

$x + y = (x_1 + y_1, \ldots, x_n + y_n)$



$a \cdot x = (ax_1, \ldots, ax_n)$

ii) Similar definition for $\mathbb{C}^n$ over $\mathbb{C}$

iii) Let $f: S \to$ [redacted] F. These functions form a vectorspace

$f_1 + f_2$ and $a \cdot f$, [redacted] when $a \in F$, are functions from S to F as well.

<u>Def</u>: Linear combination: $a_1 x_1 + \cdots + a_k x_k$ is a linear compination of $x_1, \ldots, x_k$.

Def: $x_1, \ldots, x_k$ are linearly dependent if there is a non-trivial linear combination (a combination where at least one $a_j \neq 0$) that is zero (θ).

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Linearly indepent $x_1, \dots, x_k$:

$a_1 x_1 + \quad + a_k x_k = \theta$

$\Rightarrow \quad a_1 = a_2 = \dots = a_k = 0$

Consider linearly dependent $x_1, \dots, x_k$:

We have $a_1 x_1 + \dots + a_k x_k = \theta$ and one a_i is nonzero.

We can assume $a_1 \neq 0$ without loss of generality.

Then $x_1 = -\left(\frac{a_2}{a_1}\right) x_2 - \left(\frac{a_3}{a_1}\right) x_3 - \dots - \left(\frac{a_k}{a_1}\right) x_k$

In other words, one of the vectors could be expressed as a linear combination of the other vectors.

A subspace of V, M, which is also linear vector space is called a linear manifold.

<u>Def</u> If we have a set $x_1, \dots, x_k$ of elements of V, we could consider all linear compination of $x_1, \dots, x_k$ and that set would

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be a linear manifold $M(x_1, \ldots, x_k)$. We say that this manifold is spanned by $x_1, \ldots, x_k$.

<u>Def</u>: The vectors $x_1, \ldots, x_k$ form a basis of linear manifold M if

i) $x_1, \ldots, x_k$ are linealy independent

ii) $x_1, \ldots, x_k$ span M.

In this case, the dimension of M is k.
<u>If</u> a linear manifold does not have a finite basis, it is called infinite dimensional

For the idea of dimension to make sense we have to show that if $x_1, \ldots, x_k$ is a basis for M and $y_1, \ldots, y_m$ is also a basis for it, then $k = m$.

The 'proof' in the book presupposes knowledge of linear equations, determinants etc. It is better to rely on the following lemma:

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Lemma: If $\{x_1, \cdots, x_k\}$ forms a basis for M and $\{y_1, \cdots, y_m\}$ are set of linearly independent vectors in M, then $m \leq k$.

This lemma would immediately imply $m = k$ if both $\{x_1, \cdots, x_k\}$ and $\{y_1, \cdots, y_m\}$ form bases for M.

$\{x_1, \cdots, x_k\}$ being a basis and $\{y_1, \cdots, y_m\}$ linearly indepe. $\Rightarrow m \leq k$

$\{y_1, \cdots, y_m\}$ being a basis and $\{x_1, \cdots, x_k\}$ being lin. indep. $\Rightarrow k \leq m$

So $k = m$.

How do we prove the lemma? By contradiction

Proof: Assum $m > k$. It is enough to restrict to the case $m = k+1$ and show that it is impossible. In case $m > k+1$, we will restrict our attention to $\{y_1, \cdots, y_k, y_{k+1}\}$.

The idea is to replace basis vectors by y_i's

We want to show. that the hybrid set.

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$\{y_1, \dots, y_j, \underbrace{x_{j+1}, \dots, x_k}\}$ forms a basis

k-j renumbered original basis vectors

The renumbering is important!

Let us use induction!

$j=0$ is easy. $\{x_1, \dots, x_k\}$ is already a basis

If it is true for j, let us see $j+1$ works

If $\{y_1, \dots, y_j, x_{j+1}, \dots, x_k\}$ is a basis

$$y_{j+1} = \mu_1 y_1 + \dots + \mu_j y_j + \mu_{j+1} x_{j+1} + \dots + \mu_k x_k$$

Since y's are linearly independent at least one coeff among $\mu_{j+1}, \dots, \mu_k$ has to be non zero. Let us renumber $x_{j+1}, \dots, x_k$ $\mu_{j+1}, \dots, \mu_k$ so that the nonzero element has the smallest index, namely $j+1$

With $\mu_{j+1} \neq 0$

$$x_{j+1} = \frac{\mu_1}{\mu_{j+1}} y_1 + \dots + \frac{\mu_j}{\mu_{j+1}} y_j - \frac{1}{\mu_{j+1}} y_{j+1} + \frac{\mu_{j+2}}{\mu_{j+1}} x_{j+2} \dots + \frac{\mu_k}{\mu_{j+1}} x_k$$

Since x_{j+1} can be expressed in terms of $\{y_1, \dots, y_{j+1}, x_{j+2}, \dots\}$ and $\{y_1, \dots, y_j, x_{j+1}, x_{j+2}, \dots, x_k\}$ is a basis, $\{y_1, \dots, y_{j+1}, x_{j+2}, \dots, x_k\}$

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[redacted] be a basis.

Pushing this replacement argument to $j=k$
$\{y_1, \dots, y_k\}$ would be a basis (all x's removed)

So $y_{k+1} = \lambda_1 y_1 + \dots + \lambda_k y_k$

$\Rightarrow$ y's are not linearly independent.

So dimension of a vector space is a well defined thing for finite dimensional vector spaces. We will indicate it by $\dim(M)$

Def: Standard basis for $\mathbb{R}^n$

$$e_1 = (1, 0, 0, \dots, 0)$$
$$e_2 = (0, 1, 0, \dots, 0)$$
$$\vdots$$
$$e_n = (0, 0, 0, \dots, 1)$$

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In general if $\{x_1, \cdots, x_n\}$ is a basis
$x = a_1 x_1 + \cdots + a_n x_n$

x is eqvt to this ordered tuple $(a_1, \cdots, a_n)$
These are called the components.

Beyond just the vector space structure, we want to introduce additional structures like norm and scalar products

Norm $\quad x \longmapsto \|x\| \geq 0$

With the properties a) $\|x\| = 0$ iff $x = \theta$
b) $\|ax\| = |a| \, \|x\|$, $\quad$ c) $\|x+y\| \leq \|x\| + \|y\|$
(Triangle inequality)

Example: $\quad x = \{\xi_1, \cdots, \xi_n\}$ in particular basis.

$$\|x\|_p = \left(\sum_{k=1}^{n} |\xi_k|^p \right)^{\frac{1}{p}}$$

Note that $p = 2$ is our Euclidean norm.

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Scalar product: $V \times V \to \mathbb{C}$
$(x, y) \mapsto \langle x, y \rangle$

Properties. $\langle x, y \rangle^* = \langle y, x \rangle$ (* is complex conjugate)
(Hermitean symmetry)

$\langle y, a_1 x_1 + a_2 x_2 \rangle = a_1 \langle y, x_1 \rangle + a_2 \langle y, x_2 \rangle$

$\langle b_1 y_1 + b_2 y_2, x \rangle = b_1^* \langle y_1, x \rangle + b_2^* \langle y_2, x \rangle$

(Bilinearity)

We might add: $\langle x, x \rangle > 0$ for $x \neq \theta$ (Positive-definiteness)*

Note that $\langle x, x \rangle$ [redacted] could be used

define a norm, where we have positive-definiteness.

A specific case $x \leftrightarrow (\xi_1, \ldots, \xi_n)$
$y \leftrightarrow (\eta_1, \ldots, \eta_n)$

$$\langle y, x \rangle = \eta_1^* \xi_1 + \eta_2^* \xi_2 + \cdots + \eta_n^* \xi_n$$

Here $\langle x, x \rangle = \sum_k |\xi_k|^2 = \| x \|_2^2$

Note that for the rest of the section $\|x\|$ is $\|x\|_2$.

[redacted] * We will mention indefinite metrics later in the context of relativity. Here $\langle x, x \rangle = 0 \Rightarrow x = \theta$. This is not true for Minkowski inner product.

Def: Vectors x, y are orthogonal if $\langle x, y\rangle = 0$

Def: Vectors $x_1, \dots, x_m$ form an orthogonal system if every pair of vectors are orthogonal. Some conventions restrict the definition to non-zero vectors.

Def: [redacted] The vectors $\phi_1, \dots, \phi_m$ form an orthonormal system if

$$\langle \phi_k, \phi_\ell \rangle = \underbrace{\delta_{k\ell}}_{\text{Kronecker delta}} \begin{cases} = 1 & \text{if } k=\ell \\ = 0 & \text{otherwise} \end{cases}$$

The extra bit here is that $\|\phi_k\| = \sqrt{\langle \phi_k, \phi_k\rangle} = \underline{1}$, namely, each vector is [redacted] a unit vector. Note that an orthonormal system is always linearly independent ($\sum_k a_k\phi_k = 0 \Rightarrow \langle \phi_\ell, \sum_k a_k \phi_k\rangle = a_\ell = 0$)

Orthonormal systems are specially useful when they are employed as bases.

If $\{\phi_1, \dots, \phi_m\}$ is an orthonormal basis of V and $x = a_1\phi_1 + \dots + a_m\phi_m$ for some $x \in V$, then $a_k = \langle \phi_k, x\rangle$ as $\langle \phi_k, x\rangle = \sum_\ell a_\ell \langle \phi_k, \phi_\ell\rangle = \sum_\ell a_\ell \delta_{k\ell} = a_k$

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