Sequence and Series of Functions

Pointwise Convergence and uniform convergence of Sequence of Functions:

Def: Consider a sequence of functions $\{f_n\}$, with

[redacted] $f_n : S \to \mathbb{C}$ (or $\mathbb{R}$)

Let $f : S \to \mathbb{C}$ (or $\mathbb{R}$)

$\{f_n\} \to f$ pointwise in S if for $\forall z \in S$, $\{f_n(z)\} \to f(z)$.

Note that the rate of convergence at different points could be different.

In other words, $\forall \varepsilon > 0$, $\forall z \in S$ [redacted] $\exists N(z)$ such that $|f_n(z) - f(z)| < \varepsilon$, whenever $n \geq N(z)$.

This [redacted] dependence on z in $N(z)$ captures the variable rate of convergence.

Let us take an example with a function [redacted] in reals.

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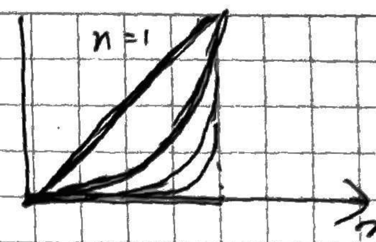

$f_n : [0,1] \to [0,1]$

$f_n : x \to x^n$

$n = 1, 2, 3, \ldots$. Define $f : [0,1) \to [0,1)$ as $f : x \to 0$.

For each $x \in [0,1]$ $x^n \to 0$ as $n \to \infty$

To make $x^n < \varepsilon$, $n > \ln \varepsilon^{-1} / \ln x^{-1}$

Define $N(x) = \text{Ceil}\left[\frac{\ln \varepsilon^{-1}}{\ln x^{-1}}\right]$

As x comes closes to 1, $N(x)$ becomes bigger and bigger for a fixed ε.

Suppose we have an N that is independent of x or z. Then we get uniform convergenc

<u>Def</u>: Consider a sequence of functions $\{f_n\}$ with $f_n : S \to \mathbb{C}$ (or $\mathbb{R}$) for each n.

Let $f : S \to \mathbb{C}$ (or $\mathbb{R}$)

$\{f_n\}$ converges to f uniformly if for $\forall \varepsilon > 0$ and $\forall z \in S$, $\exists N \in \mathbb{N}$ s.t. $n > N \Rightarrow |f_n(z) - f(z)| < \varepsilon$.

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Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2} \sin n^2 \pi x$

The series is absolutely convergent pointwise.
We can define a function $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin n^2 \pi x$
$\forall x \in \mathbb{R}$.

$$\left| f_n(x) - f(x) \right| < \sum_{m=n+1}^{\infty} \frac{1}{m^2} |\sin m^2 \pi x|$$

$$\leq \sum_{m=n+1}^{\infty} \frac{1}{m^2} = \frac{1}{(n+1)^2} + \frac{1}{n+1} < \frac{2}{n+1}$$

[use tools from integral test]

Thus, [redacted] given $\varepsilon > 0$, choose
$N = \mathrm{Ceil}\left[\frac{2}{\varepsilon}\right]$. So $n > N \Rightarrow n > \frac{2}{\varepsilon}$

$\Rightarrow n+1 > \frac{2}{\varepsilon} \Rightarrow \varepsilon > \frac{2}{n+1} \Rightarrow \varepsilon > |f_n(x) - f(x)|$

Thm: If $\{f_n\}$ is a sequence of continuous functions which uniformly converges to

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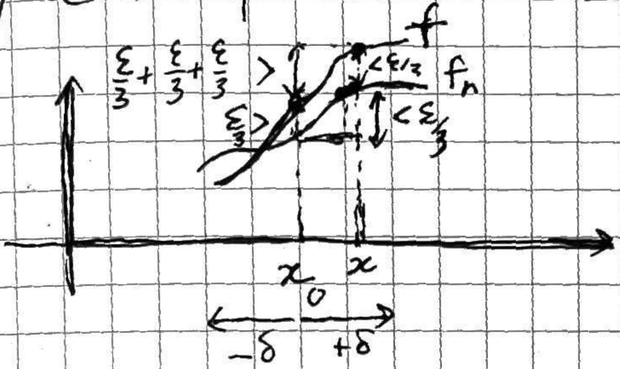
f, then f is continuous.

Proof idea!



$$\begin{aligned} &|f(x) - f(x_0)| \\ &\leq |f(x) - f_n(x)| \\ &+ |f_n(x) - f_n(x_0)| \\ &+ |f_n(x_0) - f(x_0)| \end{aligned}$$

With δ small enough and N large enough, we can guarantee each part is less ~~than~~ $\varepsilon/3 \Rightarrow |f(x)-f(x_0)| < \varepsilon$.

We will discuss some other concepts of convergence that is more relaxed. One such concept is weak convergence.

Imagine C is a vector space (more on it later) "well-behaved" functions, often called test functions, and let $\int f_n(x) g(x)\, dx$ make sense for a sequence $\{f_n\}$ and every $g \in C$.

If for every g $\left\{\int f_n(x) g(x)\, dx\right\} \to \int f(x) g(x)\, dx$ for some f, then $\{f_n\}$ converges to f weakly.

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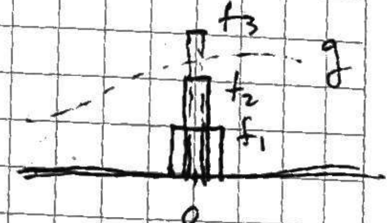
For example, $\int_0^{2\pi} \sin nx \, g(x)\,dx \to 0 = \int_0^{2\pi} 0 \, g(x)\,dx$
for any [redacted] integrable g. However the function $x \mapsto \sin nx$ [redacted] does not pointwise converge to zero.
If C was the Hilbert space (more about it later), $L^2([0,2\pi])$ we can find such nontrivial examples.

This idea could be extended to objects beyond functions. Think of $\int f(x)\,g(x)\,dx$ as a linear map $L_f : C$ [redacted] $\longmapsto R$.

For a suitable class of test functions C, we could ask what does these functions converge to

$$f_n(x) = n \qquad |x| < \frac{1}{2n}$$
$$= 0 \qquad \text{otherwise}$$



For "smooth" g, we can argue that $\int f_n(x)\,g(x)\,dx \to g(0)$. Note that that is a linear map from the function space C to a number.

This map is related to the so-called δ function but δ function is not really a function.

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$L_\delta(g) = g(0)$

Similarly we could define $L_{\delta'}(g) = -g'(0)$

etc. etc. $L_{\delta^{(n)}}(g) = (-1)^n g^{(n)}(0)$

Proper formulation of the last idea leads to the theory of distributions.

Infinite series of functions, Power series

Infinite series $\sum_k f_k(z)$ is associated with sequence partial sums $S_n(z) = \sum_{k=0}^{n} f_k(z)$. Similar definitions for absolute convergence.

Important case, power series:

$$S(z) = \sum_{n=0}^{\infty} a_n z^n$$

We should think of it as sequence of polynomials defined on $\mathbb{C}$.

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Thm: Let $S(z) = \sum_{n=0}^{\infty} a_n z^n$ be a power series, with $\alpha_n = \sqrt[n]{|a_n|}$.

If $\alpha = \limsup_{n\to\infty} \alpha_n$ then

(i) If $\alpha = 0$ the series converges absolutely for any $z \in \mathbb{C}$.

(ii) If $\alpha = +\infty$ then $S(z)$ is divergent for any $z \neq 0$

(iii) If $0 < \alpha < \infty$, then $S(z)$ is absolutely convergent for $|z| < r = \alpha^{-1}$. It converges uniformly for $|z| \leq \rho$ for any positive ρ less than r.

$S(z)$ is divergent for $|z| > r$.

Proof idea: Apply root test. $|a_n z^n|^{\frac{1}{n}} = \alpha_n |z|$

lim sup gives $\alpha|z|$.

The condition $\alpha|z| \lessgtr 1$ leads to all these consequences.

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Try examples like

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad \cos z, \sin z \longleftarrow r = \infty$$

$$(1+z)^n = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n \qquad \binom{\alpha}{n} \equiv \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$$

$$r = 1$$

$$\ln(1+z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n}$$

$\tan^{-1}(z)$ expansion.

$\sum_{n=0}^{\infty} n!\, z^n$ converges nowhere other than $z=0$

Could such an expansion ever be useful,

the answer, surprisingly, is yes!

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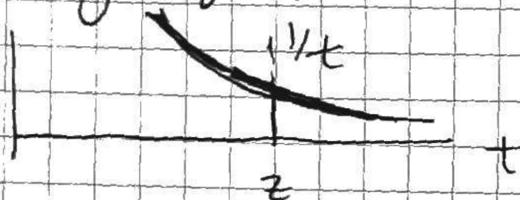
Asymptotic series

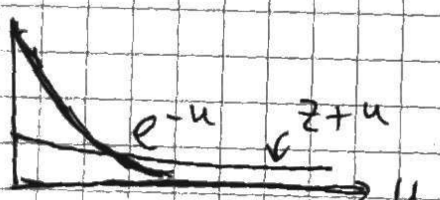
Consider $E_1(z) \equiv \int_z^\infty \frac{e^{-t}}{t}\,dt$.

For [redacted] convenience, you could think of z being large positive number



[redacted] The negative exponential changes fast compare to $1/t$.

$$E_1(z) = e^{-z}\int_0^\infty \frac{e^{-u}}{z+u}\,du \qquad t = z+u$$



The first approximation is $e^{-z}\int_0^\infty \frac{1}{z}\, e^{-u}\,du = \frac{e^{-z}}{z}$

The effect of small variation could be calculated by expanding $\frac{1}{z+u} = \frac{1}{z} - \frac{u}{z^2} + \frac{u^2}{z^3} + \cdots$

Formally $E_1(z) = e^{-z}\int_0^\infty \frac{e^{-u}\,du}{z+u} \overset{??}{=} e^{-z}\int_0^\infty du\, e^{-u}\sum_{n=0}^{\infty}\frac{(-u)^n}{z^{n+1}}$

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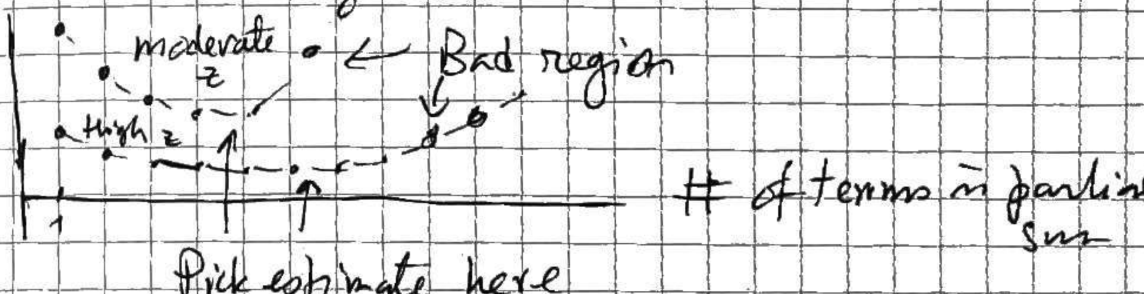
PROJECT

$$\overset{??}{=} \frac{e^{-z}}{z}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^n}\int_0^{\infty} e^{-u}u^n\,du = \frac{e^{-z}}{z}\sum_{n=0}^{\infty}\frac{(-1)^n n!}{z^n}$$

Oops!

$$= \frac{e^{-z}}{z}\left(1-\frac{1}{z}+\frac{2}{z^2}-\frac{6}{z^3}+\cdots\right)$$

For z large adding more terms help up to a point



Formal def needs Big-O little-o notation

Def: $f(z) = O[g(z)]$ as $z \to z_0$ if there is some positive M s.t. $|f(z)| \leq M|g(z)|$ in some neighborhood of z_0.

$f(z) = o[g(z)]$ if $\lim_{z \to z_0} \frac{f(z)}{g(z)} = 0$

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One can extend these definitions to $z \to \infty$ with the neighborhood of ∞ defined by $|z| > R$.

Def $\{f_n(z)\}$ is an asymptotic sequence for $z \to z_0$ if for each n

$$f_{n+1}(z) = o[f_n(z)] \quad \text{as } z \to z_0$$

Example $\{z^n\}$ as $z \to 0$

$\left\{\frac{1}{z^n}\right\}$ as $z \to \infty$

Def. If $\{f_n(z)\}$ is an asymptotic seque for $z \to z_0$ then $f(z) \sim \sum_{n=1}^{N} a_n f_n(z)$ is an asymptotic expansion if $f(z) - \sum_{n=1}^{N} a_n f_n(z) = o[f_N(z)]$

The formal series $\sum_{n=1}^{\infty} a_n f_n(z)$ is called the asymptotic series.

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In our example with $E_1(z)$ as $z \to \infty$

$f_N(z) = (-1)^N N! \dfrac{e^{-z}}{z^{N+1}}$

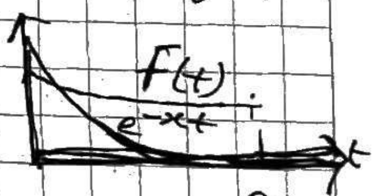
The error is order [redacted] $(N+1)! \dfrac{e^{-z}}{z^{N+2}}$

When $|z| \gg N$ $\quad \left| \dfrac{f - \sum_{n=1}^{N} a_n f_n}{f_N(z)} \right| \simeq \dfrac{N+1}{|z|} \to 0$ as $|z| \to \infty$

making it an asymptotic series

[redacted] The sharp fall of exponential could be used to create asymptotic series often

Let $J(x) = \int_0^a F(t)\, e^{-tx}\, dt$



Watson's lemma says if F is integrable on $[0,a]$

and $F(t) \sim t^b \sum_{n=0}^{\infty} c_n t^n$

then $J(x) \sim \sum_{n=0}^{\infty} c_n \dfrac{\Gamma(n+b+1)}{x^{n+b+1}}$. $\quad \left(\Gamma(\xi+1) = \int_0^{\infty} t^{\xi} e^{-t} dt \right)$

Essentially one can do a term by term integration

This has a lot of applications to Laplace integral of the form $I(z) = \int_0^a f(t)\, e^{z h(t)}\, dt$ for z large !

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