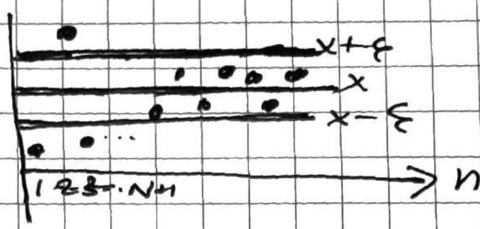
In particular we have

<u>Prop.</u>: $\{x_n\}$ converges to $x \in \mathbb{R}$ iff $x = \limsup_{n\to\infty} x_n = \liminf_{n} x_n \in \mathbb{R}$

Proof sketch:

'Only if' first!



For any $\varepsilon > 0$

$x - \varepsilon < x_n < x + \varepsilon$ when $n > N$.

So $x - \varepsilon \le v_N \le u_N \le x + \varepsilon$

$\Rightarrow x - \varepsilon \le \liminf_{n\to\infty} x_n \le \limsup_{n\to\infty} x_n \le x + \varepsilon$ for arbitrary $\varepsilon > 0$

$\Rightarrow \liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = x$

'If' bit! Let $\limsup_{n\to\infty} x_n = u$, $\liminf_{n\to\infty} = l$, $u = l = x \in \mathbb{R}$

For any $\varepsilon > 0$, show that $\exists N_1 \in \mathbb{N}$ s.t.

$x_n < u + \varepsilon$ whenever $n > N_1$

For the same ε, show that $\exists N_2 \in \mathbb{N}$, s.t.

$x_n > l + \varepsilon$ whenever $n > N_2$

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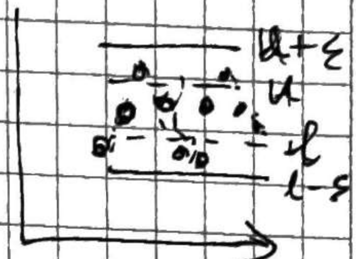
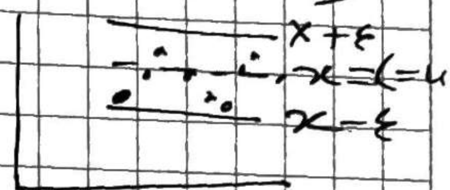
When $n > \max\{N_1, N_2\}$

$l-\varepsilon < x_n < u+\varepsilon$



But $l=u=x$

So $x-\varepsilon < x_n < x+\varepsilon$



So $\{x_n\}$ converges to x. QED.

For the next thing we need triangle inequality $|x+y| \leq |x|+|y|$

For real cauchy sequences, we know that for every $\varepsilon>0$ there is an $N\in\mathbb{N}$ s.t.
$m,n>N \Rightarrow |x_m - x_n| < \varepsilon$.

<u>Thm</u>: A real sequence is Cauchy if and only if it is convergent

Only if: For $\varepsilon>0$ choose N s.t. $n>N$ $|x_n - x| < \frac{\varepsilon}{2}$

So, if $m,n>N$ $|x_m - x_n| = |(x_m - x)-(x_n - x)|$
$\leq |x_m - x| + |x_n - x| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$

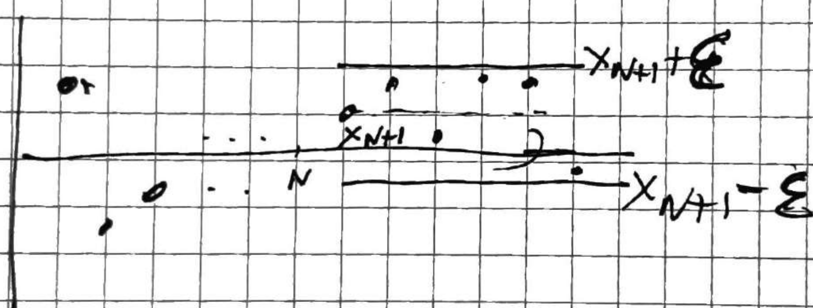
So $|x_m - x_n| < \varepsilon \Rightarrow$ Cauchy!

'If': Assume $\{x_n\}$ is cauchy.

Choose any $\varepsilon > 0$. We are guaranteed that there is an N s.t. $|x_m - x_n| < \varepsilon$ for $m, n > N$.

In particular $|x_n - x_{N+1}| < \varepsilon$ for all $n > N$. So $u_N \leq x_{N+1} + \varepsilon$

Similarly $l_N \geq x_{N+1} - \varepsilon$



So $0 \leq u_N - l_N \leq 2\varepsilon$

$$\Rightarrow 0 \leq \limsup_{n\to\infty} x_n - \liminf_{n\to\infty} x_n \leq 2\varepsilon$$

Since this is true for arbitrary ε, the limsup and lim inf are the same $\Rightarrow$ Convergent.

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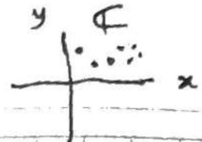

Now that we know [redacted] real Cauchy sequences are convergent and vice versa, what can we say about complex Cauchy sequences?

The idea is that we are going to relate "Cauchy"-ness ~~or~~ convergence of a complex sequence with same properties for the sequences of real and imaginary parts.

Let $\{z_n\}$ be $\{x_n+iy_n\}$ with $z_n\in\mathbb{C}$ and $x_n, y_n\in\mathbb{R}$.

Thm. $\{z_n\}$ converges iff $\{z_n\}$ is Cauchy.

If $\{z_n\}$ converges to $z = x+iy$, $x, y\in\mathbb{R}$, then for every $\varepsilon>0$ $\exists N\in\mathbb{N}$ s.t. $n>N \Rightarrow |z_n - z| < \varepsilon$

$$\Rightarrow \varepsilon > \sqrt{(x_n-x)^2+(y_n-y)^2} \Rightarrow |x_n - x| < \varepsilon \text{ and } |y_n - y| < \varepsilon$$

Thus $\{x_n\}\to x$ and $\{y_n\}\to y$. We therefore know $\{x_n\}$ and $\{y_n\}$ are Cauchy.

Now choose $\varepsilon>0$. $\exists N_1, N_2\in\mathbb{N}$ s.t. when $m,n>N_1$ $|x_m-x_n|<\frac{\varepsilon}{2}$ and $m,n>N_2$, $|y_m-y_n|<\frac{\varepsilon}{\sqrt{2}}$. Then, when $m,n > N = \max(N_1,N_2)$

$$|z_m - z_n| = \sqrt{(x_m-x_n)^2+(y_m-y_n)^2} < \sqrt{\frac{\varepsilon^2}{2}+\frac{\varepsilon^2}{2}} = \varepsilon$$

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Thus $\{z_n\}$ is Cauchy.

If $\{z_n\}$ is Cauchy, it is easy to show $\{x_n\}, \{y_n\}$ are Cauchy [redacted] (using $|z_m - z_n|$ being larger than $|x_m - x_n|$ and $|y_m - y_n|$).

Then $\{x_n\}, \{y_n\}$ are convergent. Let $\{x_n\} \to x$ and $\{y_n\} \to y$. Then for $\varepsilon > 0$, we can have N_1, s.t. $n > N_1 \Rightarrow |x_n - x| < \frac{\varepsilon}{\sqrt{2}}$ and N_2, s.t. $n > N_2 \Rightarrow |y_n - y| < \frac{\varepsilon}{\sqrt{2}}$. As a result, if $n > \max(N_1, N_2)$,

$$|z_n - (x+iy)| = \sqrt{(x_n - x)^2 + (y_n - y)^2} < \sqrt{\frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2}} = \varepsilon.$$

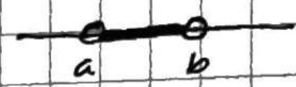
So, $\{z_n\}$ converges to $z = x + iy$.

You can easily generalize this argument for sequences for vectors in $\mathbb{R}^k$ [redacted], with convergence [redacted] and Cauchy property defined in terms of a norm, especially the euclidean one: $\|\mathbf{x}\| = \sqrt{a_1^2 + \cdots + a_k^2}$, generalizing $|x|$ and $|z|$. $\mathbf{x} = (a_1, \cdots, a_k) \in \mathbb{R}^k$.

Sets of Real and Complex Numbers:

Open and Closed sets.

Open interval: $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$



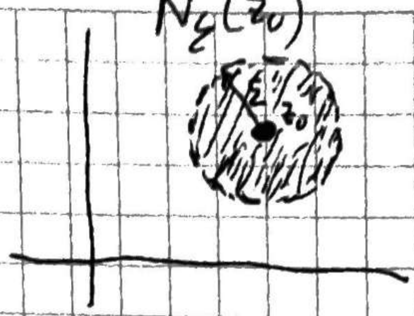
closed interval: $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$

Semi open/closed: e.g. $(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$

A neighborhood of $x_0 \in \mathbb{R}$ is an open interval containing x_0:

An ε-neighborhood $N_\varepsilon(x_0) = \{x \in \mathbb{R} \mid |x - x_0| < \varepsilon\}$

Obvious extension to complex numbers for $z_0 \in \mathbb{C}$: $N_\varepsilon(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < \varepsilon\}$



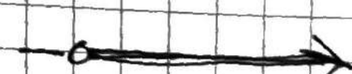
Important: The bounding circle is excluded. It is an open disc!

The textbook wants to define $N_\varepsilon(z_0)$ without the point z_0. I would not do that. As a result my definition of limit point would change slightly.

<u>Def</u>: A set $S \subset \mathbb{R}$ (or $S \subset \mathbb{C}$) is open if for every $x \in S$, there is an ε-neighborhood of x (for some $\varepsilon > 0$) $N_\varepsilon(x)$ which lies entirely [redacted] in S.

In other words, $N_\varepsilon(x) \subset S$.

Examples in $\mathbb{R}$:



Nonexamples:

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Notebook No.

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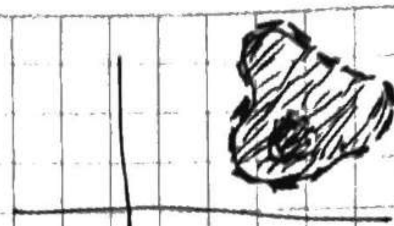
PROJECT

Examples in $\mathbb{C}$!



$\{z \in \mathbb{C} \mid |z| < 1\}$

$\{z \in \mathbb{C} \mid |z| > 1\}$

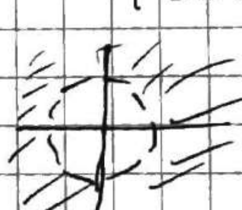
Def: Complements of open sets define closed sets.

$\mathbb{R}$ example: Open set $(-\infty, a) \cup (b, \infty)$

complement

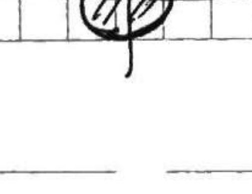
$[a, b]$

$\mathbb{C}$ example: Open set $\{z \in \mathbb{C} \mid |z| > 1\}$



Complement

Closed set



$\{z \in \mathbb{C} \mid |z| \leq 1\}$

Closed disc!

The circle $|z| = 1$ is in!

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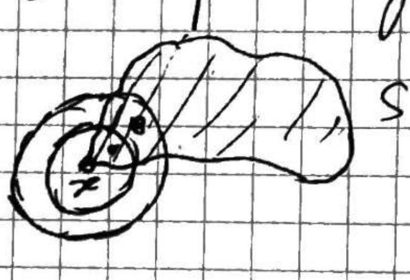
Def: The point x is a limit point of a set S if every ε-neighborhood of x contains one point of S other than x



You can replace [redacted] ε-neighborhood by [redacted] open neighborhood, namely an open set containing x.

Proposition: Every set (in $\mathbb{R}$ or $\mathbb{C}$) is closed iff it contains all its limit points.

Proof sketch: If $x \notin S$ and S contains [redacted] all limit points

$\Rightarrow \exists\, \varepsilon > 0$ s.t. $N_\varepsilon(x) \cap S = \phi$

That means $N_\varepsilon(x) \subset S^c \Rightarrow S^c$ must be open

$\Rightarrow S$ is closed.

The converse [redacted] runs the argument in the opposite direction

Bolzano-Weirstrass theorem: Every [redacted] bounded infinite set in $\mathbb{R}$ or $\mathbb{C}$ has at least

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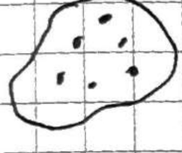
one limit point. (The statement in the book)

Sometimes this is stated as every bounded sequence has a convergent subsequence.

Essentially, the intuition is that in a "finite" size" region, if you put infinitely many points, somewhere it has got to bunch up.

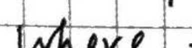


If a seq has finitely many values, one is repeated infinitely many times

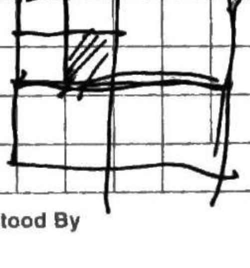
If the seq has infinitely many values, hard to put them all at some ~~fixed~~ min distance from each other.

Proof idea: Nested interval/boxes

Where is my infinity?



Where - - - ?



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Def: A subset S is dense in T if for any $x \in T$, and any open neighborhood O of x, there is a point $y \in S$ that is in O.

Example: $\mathbb{Q}$ and $\mathbb{R}$

Now, let us move onto some real examples of series

Ex I: Geometric series:

$1 + z + z^2 + z^3 + \dots$

Partial sum $S_n = 1 + z + \dots + z^{n-1} = \frac{1-z^n}{1-z}$

for $z \neq 1$.

Let us propose $1/(1-z)$ as the limit. The difference is $\left| S_n - \frac{1}{1-z} \right| = \frac{|z|^n}{|1-z|}$.

If $|z| < 1$, we can control it and show that for large enough n, this difference can be made small.

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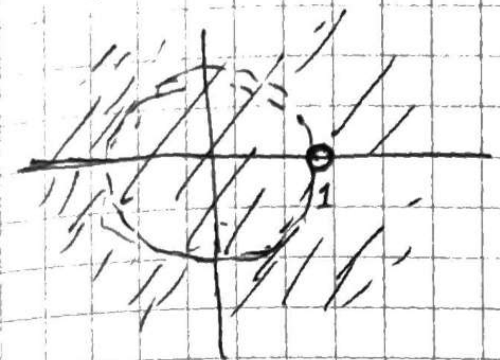
[redacted]

BTW, the function [redacted] $f: \mathbb{C}-\{1\} \to \mathbb{C}$

$f(z)$ [redacted] $= \frac{1}{1-z}$ if defined even when $|z|>1$

[redacted] and for $|z|=1$, but $z \neq 1$



The series converges only on the open set $\{z\in\mathbb{C} \mid |z|<1\}$.

We will understand this phenomenon better when we do complex analysis.

Ex 2: Riemann ζ function / Dirichlet series

Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^z}$.

Let us write $z = \sigma + i\tau$

$$\sum_{n=1}^{\infty} \frac{1}{n^{\sigma+i\tau}} = \sum_{n=1}^{\infty} \frac{e^{i\tau \ln n}}{n^{\sigma}}$$

To show absolute convergence, we consider

$$\sum_{n=1}^{\infty} \frac{1}{n^{\sigma+i\tau}} = \sum_{n=1} \frac{|e^{i\tau \ln n}|}{n^{\sigma}} = \sum_{n=1}^{\infty} \frac{1}{n^{\sigma}}$$

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Let us show that the series is divergent for $\sigma \leq 1$ and convergent for $\sigma > 1$. One could use the integral test which we will talk about soon. Instead let us do it with inequalities.

$$\sum_{n=1}^{\infty} \frac{1}{n^\sigma} \rightarrow$$ partial sums that are monotonically increasing

Only question is whether it is bounded or not.

Define the partial sum as a function $S_N(\sigma) = \sum_{n=1}^{N} \frac{1}{n^\sigma}$

Organize the series this way

$$\frac{1}{1^\sigma} + \frac{1}{2^\sigma} + \frac{1}{3^\sigma} + \frac{1}{4^\sigma} + \frac{1}{5^\sigma} + \frac{1}{6^\sigma} + \frac{1}{7^\sigma} + \frac{1}{8^\sigma} + \cdots$$

$$\boxed{\frac{1}{1^\sigma}}$$

$$\underbrace{\quad \frac{1}{2^\sigma}}$$

$$\underbrace{\frac{1}{3^\sigma} + \frac{1}{4^\sigma}}$$

$$\underbrace{\frac{1}{5^\sigma} + \frac{1}{6^\sigma} + \frac{1}{7^\sigma} + \frac{1}{8^\sigma}}$$

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Case I
$\sigma \leq 1$

$$\frac{1}{5^\sigma}+\frac{1}{6^\sigma}+\frac{1}{7^\sigma}+\frac{1}{8^\sigma} \geq \frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8} = \frac{4}{8} = \frac{1}{2}$$

In general $S_{2^m}(\sigma) - S_{2^{m-1}}(\sigma) = \sum_{n=2^{m-1}+1}^{2^m} \frac{1}{n^\sigma} \geq \frac{2^{m-1}}{2^m} = \frac{1}{2}$

$$S_{2^m}(\sigma) \geq \underbrace{\frac{1}{2}+\cdots+\frac{1}{2}}_{m \text{ times}} = \frac{m}{2}$$

So $\{S_n(\sigma)\}$ is not bounded above
$\sum_{n=1}^{\infty} \frac{1}{n^\sigma}$ diverges

Case II
$\sigma > 1$

$$\frac{1}{5^\sigma}+\frac{1}{6^\sigma}+\frac{1}{7^\sigma}+\frac{1}{8^\sigma} \leq \frac{1}{4^\sigma}+\frac{1}{4^\sigma}+\frac{1}{4^\sigma}+\frac{1}{4^\sigma} = \frac{1}{4^{\sigma-1}}$$

In general $S_{2^m}(\sigma) - S_{2^{m-1}}(\sigma) = \sum_{n=2^{m-1}+1}^{2^m} \frac{1}{n^\sigma} \leq \frac{2^{m-1}}{(2^{m-1})^\sigma} = \frac{1}{2^{(m-1)(\sigma-1)}}$

$$S_{2^m}(\sigma) \leq 1 + \frac{1}{[illegible]^{\sigma-1}} + \frac{1}{2^{\sigma-1}} + \cdots + \frac{1}{2^{(m-1)(\sigma-1)}} = 1 + \frac{1}{1-(\frac{1}{2})^{\sigma-1}}$$

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So $\{S_n(\sigma)\}$ is bounded above. $\sum \frac{1}{n^\sigma}$ is convergent.