PROJECT

Linear Operators on Hilbert Spaces

Some Hilbert space Subtleties

Ex: Let $\phi_1, \phi_2, \ldots,$ be a complete orthonormal system for [redacted] the Hilbert space H.

Define: A by $A\phi_n = \lambda_n \phi_n$

For any finite combination of $\{\phi_n\}$, A's action is defi...

If $x = \sum_n a_n \phi_n$ $\quad Ax = \sum_n \lambda_n a_n \phi_n$

$$\left.\begin{array}{l} \text{If } \sum_n |\lambda_n a_n|^2 = +\infty \\ \quad \text{while } \sum_n |a_n|^2 < +\infty \end{array}\right\} Ax \text{ is not defined for } x$$

member of Hilbert space.

Concretely $\lambda_n = n$ and $a_n = \frac{1}{n}$

In general A is defined only on a linear manifold $D_A \subset H$

If A' is defined on $D_{A'}$ with $D_{A'} \supset D_A$ and $A'x = Ax$ for $x \in D_A$ then A' is an extension of A.

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For the example with $\lambda_n = n$
Ax is defined iff $x = \sum_n a_n \phi_n$

and [redacted] $\sum_n n^2 |a_n|^2 < +\infty$.

That condition defines D_A.

Many operator's we see in quantum mechanics has to be defined on domains D_A which are proper subsets of $\mathcal{H}$. Examples, $\hat{x}$, $\frac{\hbar}{i}\frac{d}{dx} = \hat{p}$ etc

These are [redacted] examples of unbounded operators.

<u>Def</u>: Bounded operators A [redacted] are defined [redacted] by the property that for any $x \in \mathcal{H}$

$$\|Ax\| \leq C\|x\| \text{ for some } C > 0.$$

The smallest such C is called the bound, $|A|$,

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Adjoint Operator

<u>Def</u> If for any y $\langle y, Ax \rangle = \langle y_*, x \rangle$ for every $x \in D_A$, then we define $y_* = A^\dagger y$

<u>Comment</u>: For this definition to work, D_A has to be dense in $\mathcal{H}$. If $D_A^\perp$ is nontrivial, we have a [redacted] $y_0 \in D_A^\perp$, $y_0 \neq$ [redacted]. Then $\langle y_0, x \rangle = 0$ for $x \in D_A$

Hence $\langle y_* + y_0, x \rangle = \langle y, Ax \rangle$

making $A^\dagger y$ ill-defined

<u>Def</u> A linear operator is self-adjoint if $D_A = D_{A^\dagger}$ and $Ax = A^\dagger x$ for $x \in D_A = D_{A^\dagger}$

For bounded operators if D_A is dense in $\mathcal{H}$ we find an extension defined over all of th

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[redacted] In general, for unbounded symmetric operators, we might have D_{A_0} such that

$$\langle y, A_0 x\rangle = \langle A_0 y, x\rangle$$

Then A_0^+ is an extension of A_0.

Sometimes written as $A_0 \subset A_0^+$

To build a self adjoint operator we need to find an extension of A of A_0, s.t.

$$A_0 \subset A = A^+ \subset A_0^+$$

with $D_{A_0} \subset D_A = D_{A^+} \subset D_{A_0^+}$.

Spectrum of a self adjoint operator

<u>Def</u> The [redacted] spectrum of an operator A consists of $\lambda \in \mathbb{C}$ so that for every $\varepsilon > 0$ there is a $\phi \in D_A$ [redacted] s.t. $\|A\phi - \lambda\phi\| < \varepsilon \|\phi\|$

The point spectrum $\Pi(A)$ corresponds to points which are eigenvalues with $A\phi = \lambda\phi$ for some $\phi \in D_A$

$\Sigma(A) - \Pi(A) = \Sigma_c(A)$ is the continuous spectrum.

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Why do we need to be careful?

Ex 1: Consider $L^2([-1,1])$ and the operator

$$Xf(x) = x f(x)$$

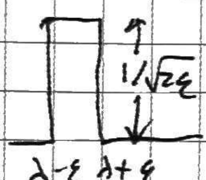
This is a bounded [redacted] operator

yet $Xf = \lambda f \Rightarrow f(x) = 0$ for $x \neq \lambda$

$f(x)$ should be $\delta(x-\lambda)$ but that is not an L^2 function.

Hence we need to find $f_\varepsilon(x) = \frac{1}{\sqrt{2\varepsilon}}$ [redacted] for $|x-\lambda| < \varepsilon$

$= 0$



$$\int \left[(x-\lambda) f_\varepsilon(x)\right]^2 dx = \varepsilon^2/3$$

$$\int |f_\varepsilon(x)|^2 dx = 1$$

So $\| X f_\varepsilon - \lambda f_\varepsilon \| = \frac{\varepsilon}{\sqrt{3}} \| f_\varepsilon \| < \varepsilon \| f \|$

Ex 2 $\quad L^2(\mathbb{R})$ with $A = -\frac{d^2}{dx^2}$

Ideally we should have e^{ikx} as an eigenfunction but that function is [redacted] not in $L^2(\mathbb{R})$.

we need to find $\psi_\varepsilon(x) = \left(\frac{\varepsilon}{\pi}\right)^{1/4} e^{ikx} e^{-\frac{\varepsilon}{2}x^2}$

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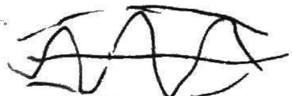

so that $\|\psi_\varepsilon\| = 1$ and it approximates the the 'almost' eigenfunction.

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The spectrum of self-adjoint operator stays on the real line.

Associated with each self adjoint operator is a resolution of identity $\{E_\lambda\}$ which projects $\mathcal{H}$ to a linear manifold M_λ. $M_\lambda \subset M_\mu$ if $\lambda < \mu$

$$\lim_{\lambda \to -\infty} E_\lambda = 0 \qquad \lim_{\lambda \to +\infty} E_\lambda = \mathbb{I}$$

$$\lim_{\varepsilon \to 0^{\pm}} E_{\lambda \pm \varepsilon} = E_\lambda^{\pm} \quad \text{exist for every } \lambda$$

If $E^+_{\lambda_0} - E^-_{\lambda_0} \neq 0$ the discontinuity corresponds to a point in the discrete spectrum

If E_λ is continuous at λ_0 but $E_{\lambda_0+\varepsilon} - E_{\lambda_0-\varepsilon} \neq 0$ λ_0 is a point of increase (continuous spectrum)

$$A = \int \lambda \, dE_\lambda \qquad \text{(Hilbert's fundamental theorem)}$$

Ex1. For finite dimensional operators we will have eigen values $\lambda_1 \cdots \lambda_n$ and corresponding Projection operators P_i

$$E_\lambda = \sum_{\substack{i \\ \text{s.t. } \lambda_i \leq \lambda}} P_i$$

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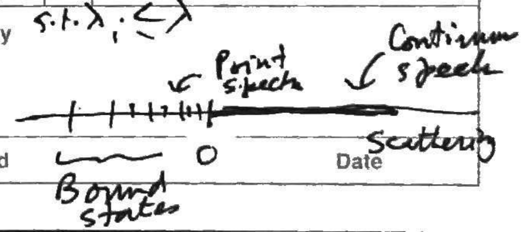
Ex2: Hydrogen atom Hamitonian



Resolvent of a self adjoint operator A

We want to define $f(A) = \int f(\lambda)\, dE_\lambda$ (e.g. $e^{iAt} = \int e^{i\lambda t} dE_\lambda$ is well defined for all $\mathcal{H}$)

In particular $R_\alpha(A) = (A - \alpha I)^{-1} = \int \frac{1}{\lambda - \alpha} dE_\lambda$

In particular, for differential operators L. we want to solve problems like

$$(L - \lambda I)\, u(x) = f(x)$$

Formally we have $(L - \lambda I)^{-1} f = I$

We want to write it as a kern $G_\lambda(x, y)$

So that $(L - \lambda I)_x G_\lambda(x, y) = \delta(x - y)$, formally.

Turns out that $G_\lambda(x, y) = G^*_\lambda(y, x)$

This is seen most easily in the case of pure discrete spectrum where
$$G_\lambda(x, y) = \sum_n \frac{\phi_n(x)\,\phi^*_n(y)}{\lambda_n - \lambda}$$

For real λ
$$G^*_\lambda(x, y) = \sum_n \frac{\phi^*_n(x)\,\phi_n(y)}{\lambda_n - \lambda} = \sum_n \frac{\phi_n(y)\,\phi^*_n(x)}{\lambda_n - \lambda} = G_\lambda(y, x)$$

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Partial Differential Equations

Linear first order PDE

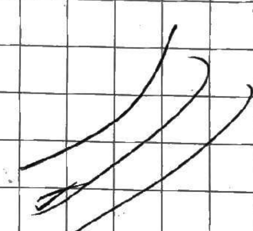
$$\sum_k v^k(x) \frac{\partial u}{\partial x^k} = f(x)$$

Such equations are solved by method of characteristics.

Solve for $\frac{dx^k}{d\lambda} = v^k(x)$ (Integral curves → characteristic)

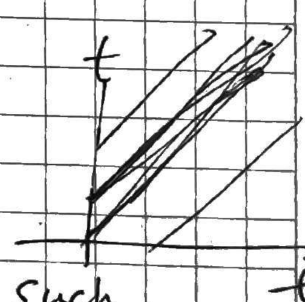
Now we get $\frac{d\, u(x(\lambda))}{d\lambda} = f(x(\lambda))$

Integrate to get u for each characteristic



Example $\frac{\partial u}{\partial t} + s\frac{\partial u}{\partial x} = 0$

Solve for $\frac{dt}{d\lambda} = 1 \quad \frac{dx}{d\lambda} = s$



Solution $t = \lambda$, $x = s\lambda + x_0$. u constant on such lines. So $u = \psi(x - st)$. Traveling wave

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Laplacian and linear second order PDE

$$\Delta u = \nabla^2 u = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)u$$

Homogeneous eqn $\nabla^2 u = 0$ (Laplace's eqn)

Inhomogeneous eqn $\nabla^2 u = -4\pi\rho$ (Poisson eqn)

Is the Laplacian self-adjoint

$$\langle v, \Delta u\rangle = \int_\Omega v^* \nabla^2 u \, d\Omega$$

$$= \int_\Omega \vec{\nabla}\cdot(v^* \vec{\nabla} u)\, d\Omega - \int_\Omega \vec{\nabla} v^* \cdot \vec{\nabla} u \, d\Omega$$

$$= \oint_S v^* \vec{\nabla} u \cdot d\vec{S} - \int \vec{\nabla} v^* \cdot \vec{\nabla} u \, d\Omega$$

$$\langle \Delta v, u\rangle = \int_\Omega (\nabla^2 v^*)\, u \, d\Omega = \int_\Omega \vec{\nabla}\cdot(u \vec{\nabla} v^*)\, d\Omega - \int \vec{\nabla} u \cdot \vec{\nabla} v^* \, d\Omega$$

So $\langle v, \Delta u\rangle - \langle \Delta v, u\rangle = \oint_S (v^* \hat{n}\cdot\nabla u - u\hat{n}\cdot\nabla v^*)\, dS$

Green's Theorem!

[redacted] A necessary condition for self adjointness [redacted] is to have the boundary term vanish

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One boundary condition that achieves that is $u(\vec{r}) = v(\vec{r}) = 0$ on S (Dirichlet Condition)

Another one is $\hat{n}\cdot\vec{\nabla}u = \hat{n}\cdot\vec{\nabla}v = 0$ on S (Neumann Condition)

More generally, we can have mixed condition like

$$\alpha(\vec{r})u(\vec{r}) + \beta(\vec{r})\hat{n}\cdot\vec{\nabla}u(\vec{r})$$
$$= \alpha(\vec{r})v(\vec{r}) + \beta(\vec{r})\,\hat{n}\cdot\vec{\nabla}v(\vec{r}) = 0$$

Let us see if we can use Green's Theorem to suggest some way of solving Laplace and Poisson equation

Let $G(\vec{r},\vec{s})$ be the Green function of given by the equation

$$\nabla^2_{\vec{r}}\, G(\vec{r},\vec{s}) = -4\pi\,\delta(\vec{r}-\vec{s})$$, with some boundary condition

[redacted]

Note that $u(\vec{r}) = \int G(\vec{r},\vec{s})\,\rho(\vec{s})\,d\Omega_s$ solves $\nabla^2_{\vec{r}}\,u(\vec{r}) = -4\pi\rho(\vec{r})$

For the case of Laplace's equation

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Use $v(\vec{s}) = G(\vec{r},\vec{s})$ and use Green's thm

$$-\int_\Omega \nabla_{\vec{s}}^2 G^*(\vec{r},\vec{s})\, u(\vec{s})\, d\Omega_{\vec{s}} + \int_\Omega G^*(\vec{r},\vec{s})\, \underbrace{\nabla_{\vec{s}}^2 u(\vec{s})}_{\text{zero}}\, d\Omega_s$$

[redacted]

$$= \int_S \left(G^*(\vec{r},\vec{s})\, \hat{n}_s \cdot \nabla_s u(\vec{s}) - u(\vec{s})\right.$$ [redacted] $$\left.\hat{n}_s \cdot \nabla G^*(\vec{r},\vec{s})\right) dS_s$$

Since $G^*(\vec{r},\vec{s}) = G(\vec{s},\vec{r})$ and $\nabla_{\vec{s}}^2 G(\vec{s},\vec{r}) = -4\pi\,\delta(\vec{s}-\vec{r})$

$$4\pi \int_\Omega \delta(\vec{s}-\vec{r})\, u(\vec{s})\, d\Omega_{\vec{s}} = \int_S \left[G^*(\vec{r},\vec{s})\, \hat{n}_s \cdot \nabla_s u(\vec{s}) - u(\vec{s})\, \hat{n}_s \cdot \nabla G^*(\vec{r},\vec{s})\right] dS_s$$

Or $$u(\vec{r}) = \frac{1}{4\pi} \int_S \left[G^*(\vec{r},\vec{s})\, \hat{n}_s \cdot \bar{\nabla}_s u(\vec{s}) - u(\vec{s})\, \hat{n}_s \cdot \nabla_s G^*(\vec{r},\vec{s})\right] dS$$

For the $\mathbb{R}^n$, $G(\vec{r},\vec{s}) = G(\vec{r}-\vec{s}) = \frac{1}{(2\pi)^n}\int e^{i\vec{k}\cdot(\vec{r}-\vec{s})}\, G(\vec{k})\, d^n\vec{k}$

Since $\int e^{ik\cdot(\bar{r}-\bar{s})}\, d^n k = (2\pi)^n \delta(\bar{r}-\bar{s})$

$\nabla_r^2 G(r,s) = -4\pi\,\delta(\bar{r}-\bar{s}) \Rightarrow -k^2 G(\bar{k}) = -4\pi$

$G(\bar{k}) = \frac{4\pi}{k^2} \Rightarrow G(\bar{r},\bar{s}) = \frac{4\pi}{(2\pi)^n}\int \frac{e^{i\bar{k}\cdot(\bar{r}-\bar{s})}}{k^2}\, d^n k$

In 3d $$G(\bar{r},s) = \frac{4\pi}{(2\pi)^3}\int \frac{e^{ik\rho\cos\theta}}{k^2}\, 2\pi k^2 \sin\theta\, d\theta\, dk \qquad \rho = |\bar{r}-\bar{s}|$$

$$= \frac{1}{\pi}\int_0^\infty \frac{e^{ik\rho} - e^{ik\rho}}{k\rho}\, dk = \frac{2}{\pi\rho}\int_0^\infty \frac{\sin u}{u}\, du$$

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$\int_{-\infty}^{\infty} \frac{\sin u}{u}\, du = \pi$

$$= \frac{1}{\rho} = \frac{1}{|\bar{r}-\bar{s}|}$$

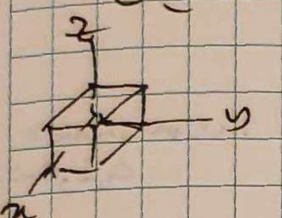
Spectrum of the Laplacian

Let u satisfy boundary conditions to make Δ selfadjoint

$$\langle u, \Delta u \rangle = -\int \vec{\nabla} u^{*} \cdot \vec{\nabla} u \, d\Omega < 0 \qquad \text{for nonzero } u.$$

If the region is bounded then the spectum becomes discrete. For example if we have

$0 \leq x \leq a$, $0 \leq y \leq b$, $0 \leq z \leq c$, with $u = 0$ on the bdry



We could try eigenfunction li:

$$u(\vec{r}) = X(x)\, Y(y)\, Z(z)$$

$$X_l(x) = \sin\frac{l\pi x}{a} \qquad Y_m(y) = \sin\frac{m\pi y}{b} \qquad Z_n(z) = \sin\frac{n\pi z}{c}$$

$$u_{lmn} = \sqrt{\frac{8}{abc}} \sin\frac{l\pi x}{a} \sin\frac{m\pi y}{b} \sin\frac{n\pi z}{c}$$

$$\lambda_{lmn} = -\left\{ \left(\frac{l\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 + \left(\frac{n\pi}{c}\right)^2 \right\}$$

If we keep periodic bdry conditions and Complex u.

$$u_{lmn}(\bar{r}) = \frac{1}{\sqrt{abc}} e^{i\left(\frac{2\pi l}{a}x + \frac{2\pi m}{b}x + \frac{2\pi n}{c}x\right)}$$

$$= \frac{1}{\sqrt{V}} e^{i\vec{k}_{lmn}\cdot\vec{r}}$$

$$\lambda = -k_{lmn}^2$$

We will often take finite volume system and then make $a,b,c \to \infty$

Note that $\frac{1}{V}\sum_{\vec{k}} g(\vec{k}) = \frac{1}{abc}\sum_{lmn} g(\vec{k}) \approx \frac{1}{abc}\int dl\,dm\,dn\; g(\vec{k})$

$$= \frac{1}{(2\pi)^3}\int dk_x\,dk_y\,dk_z\; g(\vec{k}) = \frac{1}{(2\pi)^3}\int d^3\vec{k}\; g(\vec{k})$$

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Time dependent PDE with Laplacian

$$\frac{\partial \chi}{\partial t} = a\nabla^2 \chi$$ Heat equation

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi = 0$$ Wave equation.

$$i\hbar\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\nabla^2\psi$$ Schrödinger eqn for free particle

$$i\hbar\frac{\partial}{\partial t}\psi = H\psi$$ In general.

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V(\vec{r})$$

These are homogeneous equations

For initial value problems, we get $u(\vec{r},0)$ (and maybe $\partial_t u(\vec{r},0)$)

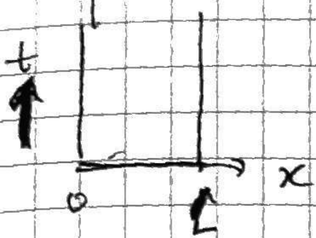
We choose "eigenfunctions" of ∇^2 (or of H) to expand

$$u(\vec{r},t) = \sum_a a_n(t)\phi_n(\vec{r})$$

Convert PDE into ODEs

Get $a_n(0)$ (maybe $\dot{a}_n(0)$) determined from initial conditions

Let us illustrate the idea for the heat equation and the wave equation. I will do it in 1+1 dim, with bdry values being 0.



$$\chi(x,0) = \sqrt{\frac{2}{L}}\sum_{n=1}^{\infty} c_n \sin\frac{\pi n x}{L}$$

$$\chi(x,t) = \sqrt{\frac{2}{L}}\sum_{n=1}^{\infty} c_n(t) \sin\frac{\pi n x}{L}$$

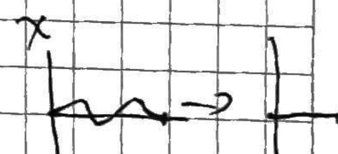
$$\frac{\partial \chi}{\partial t} = \sqrt{\frac{2}{L}}\sum_n \dot{c}_n(t) \sin\frac{\pi n x}{L} \qquad a\nabla^2 \chi = a\sqrt{\frac{2}{L}}\sum_n \left\{-\left(\frac{\pi n}{L}\right)^2\right\} c_n(t) \sin\frac{\pi n x}{L}$$

Get coeffs $\quad \dot{c}_n(t) = -a\left(\frac{\pi n}{L}\right)^2 c_n(t)$

$$c_n(t) = e^{-a\frac{\pi^2 n^2}{L^2}t}\, c_n(0)$$

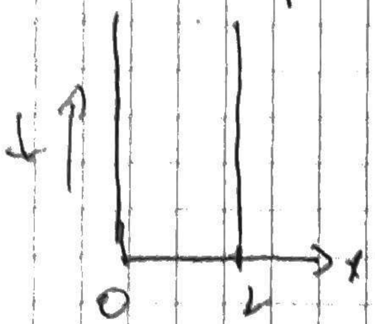
So $\chi(x,t) = \sqrt{\frac{2}{L}}\sum_{n=1}^{\infty} e^{-\frac{\pi^2 n^2 a}{L^2}t}\, c_n(0) \sin\frac{\pi n x}{L}$

In long time, $\quad \chi(x,t) \to 0$



Temperature even out. Timescale $\frac{L^2}{\pi^2 a}$

Wave equation



$$\phi(x,t) = \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} C_n(t) \sin \frac{n\pi}{L} x$$

Go through the same manouver

[redacted] $$\ddot{C}_n(t) = -\frac{c^2 n^2 \pi^2}{L^2} C_n(t)$$

$$C_n(t) = A_n \cos\left(\frac{\pi n c}{L} + \delta_n\right)$$

$$= C_n(0) \cos \frac{\pi n c}{L} t + \frac{L}{\pi n c} \dot{C}_n(0) \sin \frac{\pi n c}{L} t$$

$$\phi(x,t) = \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} A_n \cos\left(\frac{\pi n c}{L} t + \delta_n\right) \sin \frac{n\pi}{L} x$$

This solution keep oscillating.

We will now discuss inhomogeneous eqn and Greens function, in 3d.

We want to solve $\left(\vec{\nabla}_r^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi(\vec{r},t) = -4\pi\rho(\vec{r},t)$

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[redacted] We will do it via Green function

$$\left(\nabla_{\vec{r}}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\vec{r}\,\vec{s}, t-u) = -4\pi\,\delta(\vec{r}-\vec{s})\,\delta(t-u)$$

We will define $\vec{\rho}$, τ as relative coordinates
$\vec{\rho}$ is really $\vec{r}-\vec{s}$, τ is really $t-u$

We choose an $L \times L \times L$ volume. $\vec{k}_{lmn} = \left(\underset{k_x}{\frac{2\pi l}{L}}, \underset{k_y}{\frac{2\pi m}{L}}, \underset{k_z}{\frac{2\pi n}{L}}\right)$

$$G(\vec{r}\vec{s}, t-u) = \sum_{\bar{k}} g_{\vec{k}}(t-u)\,\frac{e^{i\bar{k}\cdot\bar{r}}}{\sqrt{V}}\,\frac{e^{-i\bar{k}\cdot\bar{s}}}{\sqrt{V}}$$

really lmn.

$$= \frac{1}{V}\sum_{\bar{k}} g_{\vec{k}}(t-u)\,e^{i k\cdot(\vec{r}-\vec{s})}$$

What we need is to expand $\delta(\vec{r}-\vec{s})$ in this basis

Since [redacted] all eigenvalues of $\delta(\vec{r}-\vec{s})$ as a operator is 1.

$$\delta(\bar{r}-\bar{s}) = \frac{1}{\sqrt{V}}\sum_{k} e^{i\bar{k}\cdot(\vec{r}-\vec{s})}$$

The differential equation for $g_{\vec{k}}(t-u)$

$$\left(-k^2 - \frac{1}{c^2}\frac{d^2}{d\tau^2}\right) g_{\vec{k}}(\tau) = -4\pi\,\delta(\tau).$$

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We will be interested in the retarded [redacted]
Green function (the causal one).

$G_{ret}(\vec{r}-\vec{s}, t-u) = 0$ if $t<u$ or $\tau < 0$

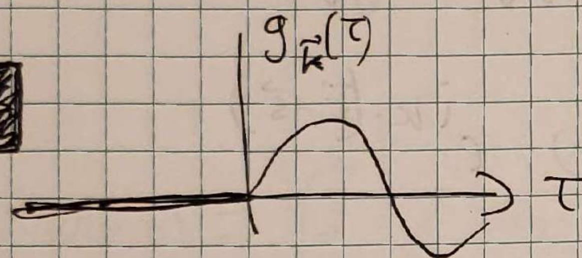
Then $g_{\vec{k}}(\tau) = 0$ for $\tau < 0$

$$\ddot{g}_{\vec{k}}(\tau) = -c^2k^2 g_{\vec{k}}(\tau) + 4\pi c^2 \delta(\tau)$$

frequency ck

This is a [redacted] harmonic oscillator with an impulse

[redacted]

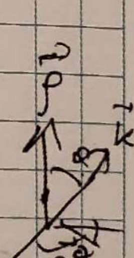


$$g_{\vec{k}}(\tau) = \frac{4\pi c}{k}\sin kc\tau$$

For $\tau > 0$

$G_{ret}(\vec{\rho}, \tau) =$ [redacted] $\frac{1}{V}\sum_{\vec{k}} \frac{4\pi c}{k}\sin kc\tau \, e^{i\vec{k}\cdot\vec{\rho}}$

$$\rightarrow \frac{4\pi c}{(2\pi)^3}\int \frac{\sin kc\tau}{k} e^{i\vec{k}\cdot\vec{\rho}} d^3k$$



$$= \frac{4\pi c}{(2\pi)^3}\int dk \int d\theta \frac{\sin kc\tau}{k} e^{ik\rho\cos\theta} 2\pi k^2 \sin\theta \, d\theta \, dk$$

[redacted]

= [redacted] $\frac{c}{\pi}\int_0^\infty \frac{e^{ik\rho}-e^{-ik\rho}}{ik\rho}\frac{\sin kc\tau}{k} k^2 dk$

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$$= \frac{2c}{\pi\rho}\int_0^\infty \sin k\rho \sin kc\tau \, dk$$

$$= \frac{c}{\pi\rho}\int_{-\infty}^{\infty} \frac{(e^{ik\rho}-e^{-ik\rho})}{2i}\frac{(e^{ikc\tau}-e^{-ikc\tau})}{2i}\, dk$$

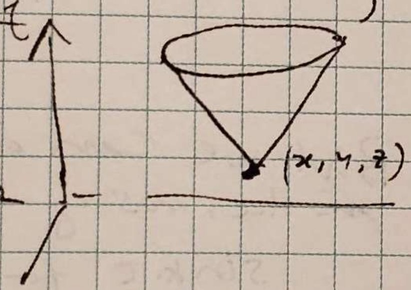
$$= \frac{c}{\pi\rho}\left(-\frac{1}{4}\right) \text{[redacted]} \left[\int_{-\infty} dk\, e^{ik(\rho+c\tau)} + \int_{-\infty} dk\, e^{-ik(\rho+c\tau)} - \int_{-\infty}^{\infty} dk\, e^{ik(\rho-c\tau)} - \int_{-\infty}^{\infty} dk\, e^{-ik(\rho-c\tau)}\right]$$

$$= \frac{c}{4\pi\rho}\left[2\times 2\pi\, \delta(\rho-c\tau) - 2\times 2\pi\, \delta(\rho+c\tau)\right]$$

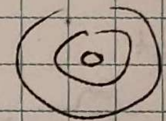
$$= \frac{c}{\rho}\left[\delta(\rho-c\tau) - \delta(\rho+c\tau)\right]$$

Since $\rho \geq 0$, $\tau > 0$, $\delta(\rho+c\tau)=0$

So $G_{ret}(\vec{\rho},\tau) = \frac{c}{\rho}\delta(\rho-c\tau)$



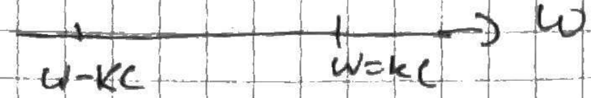
2+1 imagination



We could have tried doing it by a 4-d Fourier transfm

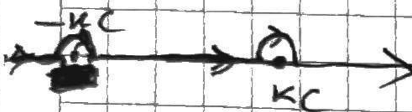
$G(\vec{\rho},\tau) \to \tilde{G}_{ret}(k,\omega)$ [redacted] satisfies $\left(\frac{\omega^2}{c^2}-k^2\right)\tilde{G}_{ret}(k,\omega) = -4\pi$

[redacted] $G_{ret}(\vec{\rho},\tau) = -\frac{4\pi c^2}{(2\pi)^3}\int \frac{e^{i\vec{k}\cdot\vec{\rho} - i\omega\tau}}{\omega^2-k^2c^2} d^3k\, d\omega$



We want $\int \frac{e^{-i\omega\tau}}{\omega^2-k^2c^2}\,d\omega$ to give the 'right' answer

$\sim \frac{\sin kc\tau}{kc}$

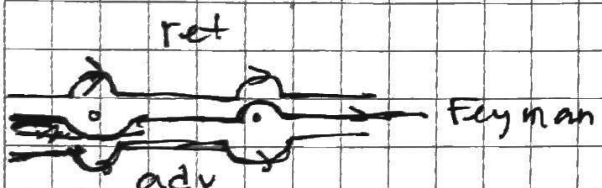


$$\int \frac{e^{-i\omega\tau}}{(\omega-kc)(\omega+kc)}\,d\omega$$

$$= (-\pi i)\,\frac{e^{+ikc\tau}}{-2kc} + (-\pi i)\,\frac{e^{-ikc\tau}}{2kc}$$

$$= +\pi i\,\frac{2i\sin kc\tau}{2kc} = -\frac{\pi\sin kc\tau}{kc}$$

$$G_{ret}(\vec{\rho},\tau) = \frac{4\pi c}{(2\pi)^3}\int_0^\infty \frac{\sin kc\tau}{k}\,e^{i\vec{k}\cdot\vec{\rho}}\,d^3k$$



But we can also think in terms of $g_{\vec{k}}(\tau)$

$\sin kc\tau$ for $\tau > 0$
or $\tau < c$

[redacted]

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