Function Spaces; Measure Theory

The most important first application of Hilbert spaces would be Fourier series. For this, we need careful definition of space of functions and the inner product on it. In turn we need a more general theory of integration via measure theory and Lebesgue integral, for that purpose.

<u>Def</u>: Let Ω be a set. A σ-algebra $\Sigma \subset 2^{\Omega}$ is a set of subsets of Ω which are closed under complement and countable union.

In other words, if $S \in \Sigma$ then $\Omega - S = S^c \in \Sigma$

If $\{S_n\}$ is a sequence in Σ then $\bigcup_{n=1}^{\infty} S_n \in \Sigma$

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Note that, since $\bigcap_n S_n = (\bigcup_n S_n^c)^c$, countable intersections also belong to Σ.

An important σ-algebra over a topological space is the Borel algebra, which is the smallest σ-algebra including the open sets.

For $\mathbb{R}$, we start with open intervals (a, b) and can construct the other members of σ-algebra by the process of taking unions and complements, and further unions etc...

Def A measure μ is a non-negative function $\mu : \Sigma \to \mathbb{R}_0^+ = \{x \in \mathbb{R} \mid x \geq 0\}$, s.t.

a) $\mu(\phi) = 0$

b) If $\{S_k\}$ [redacted] is a countable [redacted] set of disjoint members of Σ [redacted], then

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$$\mu\left(\bigcup_{k=1}^{\infty} S_k\right) = \sum_{k=1}^{\infty} \mu(S_k)$$

This property is called countable additivity.

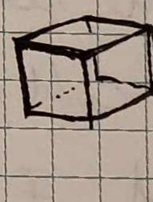
Ex 1: Think of probabity [redacted] distributions

Ex 2: The Lebesgue measure on $\mathbb{R}$ is special one which [redacted] defined on the Borel algebra on $\mathbb{R}$ in the usual topology,

$$m((a,b)) = b-a.$$

For $\mathbb{R}^n$, one could take open boxes

$$B = (a_1, b_1) \times \cdots \times (a_n, b_n)$$



$$m(B) = \prod_{k=1}^{n} (b_k - a_k)$$

Note that a singleton [redacted] $\{x \in \mathbb{R}\} = \bigcap_{k=1}^{\infty} (x-\frac{1}{k}, x+\frac{1}{k})$ is a measurable set. If $S_1 \subset S_2$ and $S_1, S_2 \in \Sigma$, $S_2 \cap S_1^c \in \Sigma$

So $m(S_1) + m(S_2 \cap S_1^c) = m(S_2) \Rightarrow m(S_2) \geq m(S_1)$. Since $\{x\} \subset \{x-\frac{1}{k}, x+\frac{1}{k}\}$. $0 \leq m(\{x\}) \leq \frac{2}{k}$ for all k. Hence $m(\{x\}) = 0$

So, countable subsets of $\mathbb{R}$ has measure zero.

For example, $m(\mathbb{Q}) = 0$.

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Now we move to functions.

Def: $f: \Omega \to X$, for X with a topology and Ω with a σ-algebra Σ, is called measurable if for any open set $U \subset X$, $f^{-1}(U)$ is in the σ-algebra, i.e. $f^{-1}(U) \in \Sigma$.

Note: We do not need to invoke a measure, yet.

We would ofte[redacted] talk of function $f: [a,b] \to \mathbb{R}$ or $f: [a,b] \to \mathbb{C}$, [redacted] with Borel algebra on $[a,b]$

It turns [redacted] out that for complex function, the real and the imaginary part has to be measurable in the real sense. Also, for a real function f we could define $f_+(x) = \max\{f(x), 0\}$ and $f_-(x) = -\min\{f(x), 0\}$.

$f = f_+ - f_-$, with f_+ and f_- both non negative. If f is [redacted] measurable, so are f_+, f_-.

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Lebesgue integral is define for nonnegative real functions by defining "step functions". In practice we define simple functions. For a measurable set S define

$$\chi_S(x) = 1 \text{ if } x \in S, \quad = 0 \quad \Big\} \text{ characteristic function of } S.$$

Def A measurable simple function σ (step function in the book) is a finite linear combination $\sum_{k=1}^{n} \sigma_k \chi_{S_k} = \sigma$ for collection of measurable sets $\{S_k\}$ and real numbers $\{\sigma_k\}$.

Define: $\int_S \sigma \, d\mu = \sum_{k=1}^{n} \sigma_k \, \mu(S_k \cap S)$ for any measurable set S and a simple measurable σ.

Def For a nonnegative real functions f the Lebesgue integral over a measurable set S is defined by

$$\int_S f \, d\mu = \sup_{\substack{0 \le \sigma \le f \\ \sigma \text{ simple and measurable}}} \int_S \sigma \, d\mu$$

Def For general real function, $\int_S f \, d\mu$ is defined if $\int_S f_+ \, d\mu$ and $\int_S f_- \, d\mu$ are defined and at least one of the integrals are finite.

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then $\int_S f \, d\mu = \int_S f_+ \, d\mu - \int_S f_- \, d\mu$

Complex integrals are defined if integrals of real & im are defined.

After this long detour, we come to $L^2(\Omega)$ spaces. This space is made of measurable functions $f: \Omega \to \mathbb{C}$ so that

$$\int_\Omega |f|^2 d\mu$$ is defined and is finite.

It is a normed space $\|f\| = \sqrt{\int_\Omega |f|^2 d\mu}$ and it has a scalar prod. $\langle f, g \rangle = \int f^* g \, d\mu$

Note that $\int_\Omega |f|^2 d\mu = 0$ does not mean [redacted] $f = 0$, but that $f \neq 0$ on a set of measure zero.

So, instead of dealing with f we need to deal with equivalence classes of f that differ[redacted] only on sets of measure zero.

We will use the same notation f to indicate the equivalence class.

$L^2(\Omega)$ is complete according to Riesz-Fischer theorem

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<u>Thm</u>: If $f_1, f_2, \dots$ is a Cauchy sequence in $L^2(\Omega)$ in the it converges to an $f \in L^2(\Omega)$.

Fourier Series

We consider function $f : \mathbb{R} \to \mathbb{C}$ where $f(t+T) = f(t)$. We could focus on the primary domain to be $[-T/2, T/2]$

and take $\phi_n(t) = \frac{1}{\sqrt{T}} e^{in\omega t}$ with $\omega = \frac{2\pi}{T}$ and $n \in \mathbb{Z}$

then $\{\phi_n\}$ form an orthogonal system on $[-T/2, T/2]$.

$$\int_{-T/2}^{T/2} \phi_n^*(t)\,\phi_m(t)\,dt = \frac{1}{T}\int_{-T/2}^{T/2} e^{2\pi i(m-n)t/T}\,dt$$

$$= 1 \quad \text{if } m = n$$

$$= 0 \quad \text{if } m \neq n$$

It turns out that this is a complete orthonormal system, and therefore

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any [redacted] $f \in \mathcal{L}^2([-\frac{T}{2}, \frac{T}{2}])$ could be expanded as $f = \sum_{n=-\infty}^{\infty} c_n \phi_n$

with $c_n =$ [redacted] $\int_{-\frac{T}{2}}^{\frac{T}{2}} \phi_n^*(t)\, f(t)\, dt$

[redacted]

To prove this one needs to know that trigonometric polynomials are [redacted] everywhere dense in L^2. Textbooks often do this in two steps: First show a version Weirstrass approximation theorem, proving that continuous functions are well approximated by [redacted] trigonometric polynomials. [redacted] Then they show that L^2 functions are well approximated by continuous functions in the L^2 sense.

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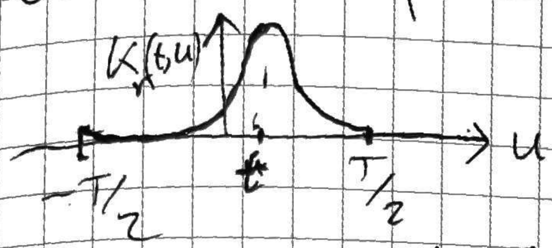
To get sense how this might work, we will try a direct approach, without proof.

Consider the function $\frac{1}{J_n}\left[\frac{1+\cos\frac{2\pi(t-u)}{T}}{2}\right]^n = K_n(t-u)$



This function (in fact, a kernel) is very peaked around $t=u$, for large n

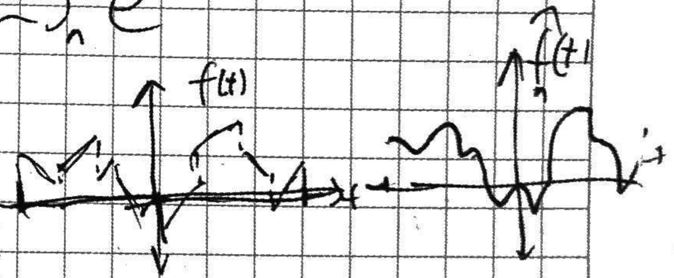
J_n is chosen so that $\int_{-T/2}^{T/2} K_n(t-u)\,du = 1$

When n is large $K_n(t-u) \approx \frac{1}{J}\left[\frac{1+1-\frac{1}{2}\frac{4\pi^2}{T^2}(t-u)^2}{2}\right]^n$

for $|t-u| \approx \frac{1}{\sqrt{n}}$

$\approx J_n^{-1} e^{-\frac{n\pi^2}{T^2}(t-u)^2}$

And $J_n^{-1} \approx \frac{1}{\sqrt{\pi}}\frac{\sqrt{n}\,\pi}{T} = \frac{\sqrt{n\pi}}{T}$



Now $f(t)$ and $\hat{f}_n(t) = \int_{-T/2}^{T/2} K_n(t-u) f(u)\,du$ could be compared. Note that $\hat{f}(t)$ could be written in terms of trigonometric polynomials. $\hat{f}_n(t)$ is a slightly smoothed

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version of f. Of course showing that (for large n) they are close in L^2 sense require more technical work.

Now we spend some time thinking about pointwise convergence of Fourier series.

For the sake of convenience, let us set

$$T = 2\pi$$

$-\pi$ π

$$\phi_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \phi_n(x)$$

$$c_n = \langle \phi_n, f \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-inx} f(x)\, dx$$

Consider the partial sum

$$f_N(x) = \sum_{n=-N}^{N} c_n e^{inx}/\sqrt{2\pi} = \frac{1}{2\pi} \sum_{n=-N}^{N} \int_{-\pi}^{\pi} e^{in(x-y)} f(y)\, dy$$

$\sum_{n=-N}^{N} e^{inu}$ [redacted] $= e^{-iNu} + e^{-i(N-1)u} + \cdots + e^{i(N-1)u} + e^{iNu}$

$$= e^{-iNu}\left(1 + e^{iu} + \cdots + e^{2Niu}\right)$$

$$= e^{-iNu} \frac{1 - e^{(2N+1)iu}}{1 - e^{iu}} = \frac{e^{i(N+\frac{1}{2})u} - e^{-i(N+\frac{1}{2})u}}{e^{iu/2} - e^{-iu/2}}$$

$$\frac{\sin(N+\frac{1}{2})u}{\sin\frac{1}{2}u} = D_N(u)$$

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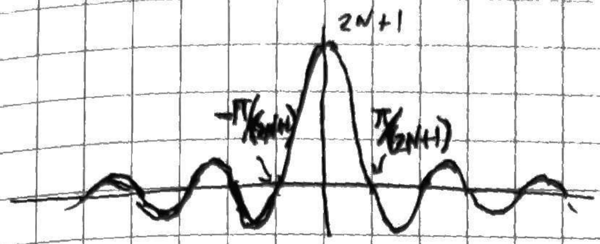
PROJECT

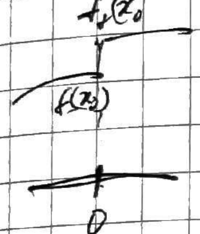
$D_N(u)$ is the [redacted] Dirichlet kernel.



$$\int_{-\pi-x}^{\pi-x} D_N(u)\,du = \sum_{n=-N}^{N}\int_{-\pi-x}^{\pi-x} e^{inu}\,du = 2\pi\sum_{n=-N}^{N}\delta_{n,0} = 2\pi$$

$$\text{So } f_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(x-y)\,f(y)\,dy = \frac{1}{2\pi}\int_{-\pi-x}^{\pi-x} D_N(u)\,f(x+u)\,du$$

We used $D_N(u) = D_N(-u)$



Use transform^n. $y = x+u$

If f is piecewise continuous and have left and right limits $f_\pm(x_0)$ at x_0

$$f_N(x_0) - \frac{1}{2}\left(f_+(x_0) + f_-(x_0)\right) = \frac{1}{2\pi}\int_{-\pi}^{0} D_N(u)\left(f(x_0+u) - f_-(x_0)\right)du$$
$$+ \frac{1}{2\pi}\int_{0}^{\pi} D_N(u)\left(f(x_0+u) - f_+(x_0)\right)du$$

These two integrals could be [redacted] rewritten

$$\Delta_\pm^N = \pm\frac{1}{\pi}\int_0^{\pi/2}\left\{\frac{f(x_0\pm 2\theta) - f_\pm(x_0)}{\sin 2\theta}\right\}\sin(2N+1)\theta\,d\theta$$ With $u = 2\theta$

If [redacted] $\dfrac{f(x_0\pm2\theta) - f_\pm(x_0)}{\sin\theta}$ is a smooth function this integral vanishes as $N \to \infty$.

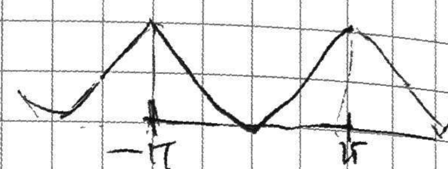
[If those functions are square integrable, the integrals are Fourier [redacted] coefficients which would go to zero as $N \to \infty$]

We can write in terms of real functions and use the orthonormal basis:

[redacted] $\frac{1}{\sqrt{2\pi}}$, [redacted] $\frac{1}{\sqrt{\pi}}\cos x, \frac{1}{\sqrt{\pi}}\sin x, \frac{1}{\sqrt{\pi}}\cos 2x, \frac{1}{\sqrt{\pi}}\sin 2x, \ldots$ etc

$$f(x) = \frac{a_0}{\sqrt{2\pi}} + \frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

Example: $f(x) = |x|$



$$f(x) = \frac{1}{\sqrt{2\pi}}a_0 + \frac{1}{\sqrt{\pi}}\sum_{m=1}^{\infty} a_m \cos mx + \underbrace{\frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty} b_m \sin mx}_{\text{[redacted]}\ 0}$$

since the function is even

$$a_0 = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}|x|\,dx = \frac{2}{\sqrt{2\pi}}\int_0^{\pi}|x|\,dx = \frac{2}{\sqrt{2\pi}}\frac{\pi^2}{2}$$

$$a_n = \frac{1}{\sqrt{\pi}}\int_{-\pi}^{\pi}|x|\cos mx\,dx = \frac{2}{\sqrt{\pi}}\int_0^{\pi} x\cos mx\,dx$$

Let us do it this way $\int_0^{\pi}\sin \mu x\,dx = \left[-\frac{\cos \mu x}{\mu}\right]_0^{\pi} = \frac{1-\cos \pi\mu}{\mu}$

Differentiate in μ: $\int_0^{\pi} x\cos(\mu x)\,dx = -\frac{1-\cos\pi\mu}{\mu^2} + \frac{\pi\sin\pi\mu}{\mu}$

$$= \frac{\pi\mu\sin\pi\mu - (1-\cos\pi\mu)}{\mu^2}$$

Now set $\mu =$ [redacted] $1, 2, 3, \ldots$

If $\mu = 2n$, $n = 1, 2, \ldots$ $\sin 2\pi n = 0$ and $1-\cos 2\pi n = 0$

So $\mu = 2n+1$, $n = 1, 2, \ldots$

$$\int_0^{\pi} x\cos(2n+1)x\,dx = \frac{\pi(2n+1)\underbrace{\sin(2n+1)\pi}_{\text{zero}} - (1-\underbrace{\cos(2n+1)\pi}_{-1})}{(2n+1)^2} = -\frac{2}{(2n+1)^2}$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\frac{\pi^2}{\sqrt{2\pi}} + \frac{1}{\sqrt{\pi}}\sum_n \frac{2}{\sqrt{\pi}}\left[-\frac{2}{(2n+1)^2}\right]\cos(2n+1)x$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)x}{(2n+1)^2}$$

Since the n-th term goes as $\frac{1}{n^2}$, this is an absolutely convergent series for all x

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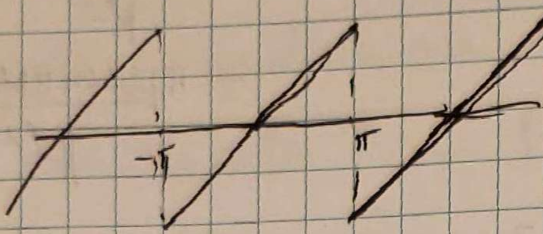
Now consider the discontinuous function $f(x) = x$ for $-\pi < x \leq \pi$



We only get contribution from $\sin nx$ here

$$f(x) = \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} x \sin nx \, dx$$

Consider the integral $\int_{-\pi}^{\pi} \cos \nu x \, dx = \frac{2 \sin \nu\pi}{\nu}$

Differentiating in ν we get

$$- \int_{-\pi}^{\pi} x \sin \nu x \, dx = -\frac{2 \sin \nu\pi}{\nu^2} + \frac{2\pi \cos \nu\pi}{\nu}$$

Now put $\nu = 1, 2, 3,$ [redacted], $\sin n\pi = 0$
$\cos n\pi = (-1)^n$

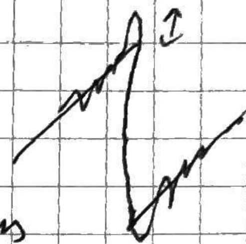
$$b_n = \frac{1}{\sqrt{\pi}} \int_{-\pi}^{\pi} x \sin nx \, dx = \frac{1}{\sqrt{\pi}} \frac{2\pi(-1)^{n-1}}{n}$$

$$f(x) = \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi}} \frac{2\pi(-1)^{n-1}}{n} \sin nx = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sin nx}{n}$$

This series is not absolutely convergent at $x \neq 0, \pm\pi$

Notice that at π, the series gives 0, the average of left and right limits.

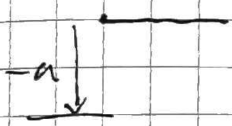
If one looks carefully at partial sums [redacted] near the discontinuity, one sees jumps



These jump stays as $N\to\infty$, but happens closer and closer to the discontinuity

This is called the Gibbs phenomenon.

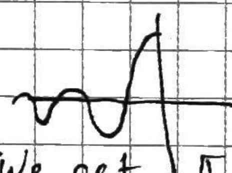
To see how it [redacted] arises, consider the simple case
$f(x)=0$ for $x\geq 0$ and $f(x)=-a$ for $x<0$



For small x $\quad f_N(x)=\frac{1}{2\pi}\int D_N($[redacted]$u)\,f(x+u)\,du$

$$= -\frac{a}{2\pi}\int_{-\pi}^{-x} D_N(u)\,du$$

$$\int_{-\pi}^{-x}\frac{\sin(N+\frac{1}{2})u}{\sin\frac{1}{2}u}\,du \quad \text{for } x=0$$



we get π

$$f_N(0) = -\frac{a}{2\pi}\pi = -\frac{a}{2}$$ as expected

What if we put $x_N=\frac{2\pi}{2N+1}$

we get a negative answer

$$\int_{-\pi}^{-\frac{2\pi}{2N+1}}\frac{\sin(N+\frac{1}{2})u}{\sin\frac{1}{2}u}\,du$$ change var to $\xi=\frac{(2N+1)u}{2}$

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$$\int_{-(N+\frac{1}{2})\pi}^{-\pi}\frac{\sin\xi}{\sin\frac{\xi}{2N+1}}\frac{2}{2N+1}\,d\xi \simeq 2\int_{\pi}^{\infty}\frac{\sin\xi}{\xi}\,d\xi \Rightarrow -\frac{a}{\pi}\int_{\pi}^{\infty}\frac{\sin\xi}{\xi}\,d\xi$$

Jump independent of N.

Fourier Transform

Let us bring back T

$$f(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} c_n e^{-\frac{2\pi i n t}{T}} \qquad \text{for } t \in \left[-\frac{T}{2}, \frac{T}{2}\right]$$

$$C_n = \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{\frac{2\pi i n t}{T}} f(t)\, dt$$

Call $\omega_n = \frac{2\pi n}{T}$, $\Delta\omega = \frac{2\pi}{T}$

$$f(t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} C(\omega_n) e^{-i\omega_n t} \Delta\omega$$

If we let $T \to \infty$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} C(\omega) e^{-i\omega t}\, d\omega$$

$$C(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} f(t)\, dt$$

The function C is the Fourier tranform of the function f.

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$$\int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t)|^2 dt = \frac{1}{T^2}\sum_n |c_n|^2 T = \frac{1}{T}\sum_n |c_n|^2$$

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} |f(t)|^2 dt = \frac{1}{T}\sum_{n=-\infty}^{\infty} |c_n|^2 \rightarrow \text{Parseval formula}$$

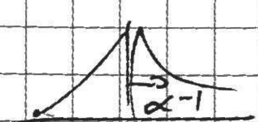
$$= \frac{1}{2\pi}\sum_n |C(\omega_n)|^2 \Delta\omega$$

When $T \rightarrow \infty$

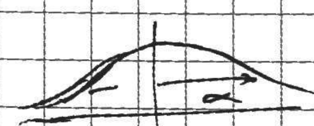
$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |C(\omega)|^2 d\omega$$

This is the Plancherel's formula

Example: $f(t) = A e^{-\alpha|t|}$



$$C(\omega) = [redacted] \frac{A\alpha}{\omega^2+\alpha^2}$$



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Convolution Theorem

Def: The convolution of f and g is defined by

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t-u)\, g(u)\, du$$

Let

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} c(\omega) e^{-i\omega t} d\omega$$

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d(\omega) e^{-i\omega t} d\omega$$

$$(f * g)(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} d\omega \; c(\omega) e^{-i\omega(t-u)} \; g(u)$$ [redacted]

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, c(\omega)\, e^{-i\omega t} \int du$$ [redacted] $$e^{i\omega u}\, g(u)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} c(\omega) d(\omega) e^{-i\omega t} \; d\omega$$

So $f * g$ has the Fourier transform which is the product of the Fourier transform $c \times d$.

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Multidimensional Fourier transform

$$f(\vec{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int e^{i\vec{k}\cdot\vec{x}} \phi(\vec{k})\, d^n k$$

$$\phi(\vec{k}) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int e^{-i\vec{k}\cdot\vec{x}} f(\vec{x})\, d^n x$$

That $f(\vec{x}) = \frac{1}{(2\pi)^n} \int e^{i\vec{k}\cdot\vec{x}} \left(\int e^{-i\vec{k}\cdot\vec{y}} f(\vec{y})\, d^n y \right) d^n k$

Formally by switching order of integration

$$= \int \left\{ \frac{1}{(2\pi)^n} \int e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\, d^n k \right\} f(\vec{y})\, d^n y$$

This is formally captured by

$$\frac{1}{(2\pi)^n} \int e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\, d^n k = \delta(\vec{x}-\vec{y})$$

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