

Infinite-dimensional Vector Spaces

In this part of the course, we revisit linear algebra, with special focus on infinite dimensional spaces, often arising from function spaces. Function spaces, Fourier series, and "convergent" of infinite series of functions ^{are} ~~are~~ important in many areas of mathematical physics, and ^{these concepts involve} ~~involve~~ infinite dimensional vector spaces, in particular, Hilbert spaces.

In our ~~earlier~~ earlier discussion, we had defined finite dimensional bases for finite dimensional vector spaces. For infinite dimensional spaces, things are a bit more complicated: we have Hamel basis, which can be defined purely algebraically and Schauder basis, which require additional structures Continued on Page providing a meaning to convergence. Read and Understood By

Def: If vector space V has a linearly independent subset B which spans V , then B is a Hamel basis of V .

Comment 1) Linear independence means if $x_1, \dots, x_m \in B$ and $a_1 x_1 + \dots + a_m x_m = \theta$, $\{a_i\}$ being appropriate scalars, then $a_1 = a_2 = \dots = a_m = 0$.

Comment 2) B spanning V means, that any $x \in V$ can be expressed as a finite linear combination of member of B , i.e. $x = a_1 x_1 + \dots + a_m x_m$ for $x_1, \dots, x_m \in B$.

Comment 3) Such a representation is unique (shown using the linear independence property of B)

Comment 4) Every vector space has a Hamel basis (proved, in general, using a non-constructive method based on Zorn's Lemma/axiom of choice)

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Comment 5) For many infinite dimensional spaces, Hamel basis is an uncountable set, making it somewhat unwieldy.

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Previously, we had defined normed vector spaces, where for every $x \in V$ we can define a nonnegative number $\|x\| \geq 0$ ("length") satisfying

a) $\|x\| = 0$ iff $x = 0$

b) $\|ax\| = |a| \|x\|$

c) $\|x+y\| \leq \|x\| + \|y\|$

(we are thinking of $a \in \mathbb{R}$ or $\mathbb{C}$)

Def: A complete normed vector space is called a Banach space.

Comment: This means if there is a sequence $\{x_n \in V\}_{n=1}^{\infty}$ which is Cauchy in the norm, it converges to some $x \in V$ in the same norm.

Def: In an ^(infinite dimensional) Banach space V , if we have an ordered sequence $\{x_n\}_{n=1}^{\infty}$, so that for any $x \in V$

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there is a unique sequence $\{a_n \in \mathbb{C}\}_{n=1}^{\infty}$ so that

$$\|x - \sum_{j=1}^n a_j x_j\| \rightarrow 0 \quad \text{as } n \rightarrow \infty, \text{ then}$$

$\{x_n\}$ is known as the Schauder basis.

Comment 1)

Schauder basis ordering is important, unlike the Hamel basis.

Comment 2) Such a basis is countable.

On function spaces, one can define norms which make it into Banach spaces. Example: L^p spaces.

We will focus on Hilbert spaces

Def: A Hilbert Space is a unitary vector space that is complete (with the ^{metric} norm induced by the scalar product).

Some textbook (including Vaughn) adds the condition that it is separable (has a countable basis). ^(in the topological sense) A consequence of this is ~~it~~ having a countable basis, in the sense of vector spaces.

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Example: Consider the space $l^2(\mathbb{C})$,
 $l^2(\mathbb{C}) = \{ (x_1, x_2, \dots) \in \mathbb{C}^\infty \mid \sum_{k=1}^{\infty} |x_k|^2 < +\infty \}$

Let $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$

$$\langle x, y \rangle = \sum_{k=1}^{\infty} x_k^* y_k$$

The R H S is absolutely convergent as
 $\left| \sum_{k=1}^N x_k^* y_k \right| \leq \sqrt{\sum_{k=1}^N |x_k|^2} \sqrt{\sum_{k=1}^N |y_k|^2}$ (Cauchy Schwarz)
 $\leq \sqrt{\sum_{k=1}^{\infty} |x_k|^2} \sqrt{\sum_{k=1}^{\infty} |y_k|^2}$

Note that $\|x\|^2 = \sum_{k=1}^{\infty} |x_k|^2$, $\|y\|^2 = \sum_{k=1}^{\infty} |y_k|^2$.

This space has a Schauder basis: $\phi_1 = (1, 0, 0, \dots)$, $\phi_2 = (0, 1, \dots)$, ...
 which is orthonormal.

Statement 1: $l^2(\mathbb{C})$ is separable.

Proof: We can consider the subset of $l^2(\mathbb{C})$ made of $x = (x_1, x_2, \dots)$
 where x_k 's have real and imaginary
 parts made of rational numbers and with
 only finitely many x_k 's are non zero.
 Such a set is countable and everywhere dense in $l^2(\mathbb{C})$.

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To set up a countable basis in the topological sense,

One can consider balls of rational radii around these points in $\ell^2(\mathbb{C})$.

Statement 2: This space is complete.

Proof: If $\{x_m\}_{m=1}^{\infty}$ is ~~Cauchy~~ Cauchy sequence, with

$$x_m = (\xi_{m1}, \xi_{m2}, \dots), \text{ then}$$

$\{\xi_{mk}\}_{m=1}^{\infty}$ is a Cauchy sequence for each $k=1, 2, \dots$

This is because $\|\xi_{mk} - \xi_{nk}\| \leq \|x_m - x_n\|^2 = \sum_{k=1}^{\infty} |\xi_{mk} - \xi_{nk}|^2$

~~Cauchy~~ $\{\xi_{mk}\}_{m=1}^{\infty}$ being Cauchy implies it converges to a limit ξ_k , for each $k=1, 2, \dots$

Consider $x = (\xi_1, \xi_2, \dots)$. We first need to show $x \in \ell^2(\mathbb{C})$. To show that,

note $\|x_m\|^2 \leq B^2$ for all $m=1, 2, \dots$ for some $B \in \mathbb{R}$

(a modification of our proof the Cauchy sequence are bounded for $\ell^2(\mathbb{C})$)

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For any N
$$\sum_{k=1}^N |\xi_k|^2 = \lim_{m \rightarrow \infty} \sum_{k=1}^N |\xi_{mk}|^2$$

Since
$$\sum_{k=1}^N |\xi_{mk}|^2 \leq \|x_m\|^2 \leq B^2 \text{ for all } m$$

So,
$$\sum_{k=1}^N |\xi_k|^2 \leq B^2 \Rightarrow \sum_{k=1}^{\infty} |\xi_k|^2 \leq B^2$$

So $x \in l^2(\mathbb{F})$.

Now, to show $x_m \rightarrow x$ as $m \rightarrow \infty$, we use the same idea.

$$\sum_{k=1}^N |\xi_{mk} - \xi_k|^2 = \lim_{n \rightarrow \infty} \sum_{k=1}^N |\xi_{mk} - \xi_{nk}|^2$$

For any $\epsilon > 0$ we can choose large enough N_0 ,

s.t. $m, n > N_0$ implies $\|x_m - x_n\| < \epsilon$.

Hence
$$\sum_{k=1}^N |\xi_{mk} - \xi_{nk}|^2 < \epsilon^2 \text{ once } m, n \geq N_0$$

Then
$$\sum_{k=1}^N |\xi_{mk} - \xi_k|^2 < \epsilon^2 \text{ once } m > N_0$$

Hence
$$\|x_m - x\|^2 \leq \epsilon^2 \text{ for } m > n$$

Using this fact we can show that $x_m \rightarrow x$ as $m \rightarrow \infty$.

Theorem: All separable infinite dimensional Hilbert space (over complex numbers) is isomorphic to $l^2(\mathbb{C})$.

Proof: If we have a everywhere dense countable set, we can use the Gram-Schmidt orthonormalization process to generate an orthonormal basis. This basis would be infinite dimensional if the Hilbert space is infinite dimensional.

Once we have this basis $\{\phi_k\}_{k=1}^{\infty}$, any vector x in the Hilbert space H , can be expressed as

$$x = \sum_{k=1}^{\infty} \xi_k \phi_k, \text{ with } \|x\|^2 = \sum_{k=1}^{\infty} |\xi_k|^2 < +\infty.$$

Also, any $(\xi_1, \xi_2, \dots) \in l^2(\mathbb{C})$ gives rise to a unique $x = \sum_{k=1}^{\infty} \xi_k \phi_k \in H$.

So $x \in H \leftrightarrow (\xi_1, \xi_2, \dots) \in l^2(\mathbb{C})$, showing

the isomorphism.

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Let $\{\phi_1, \phi_2, \dots\}$ be an orthonormal system, meaning $\langle \phi_k, \phi_l \rangle = \delta_{kl}$. Like in finite dimensional system, we will call a system complete if $\langle \phi_k, x \rangle = 0$ for all k implies $x = 0$.

Thm: The orthonormal system $\phi_1, \phi_2, \dots$ in a Hilber space $\mathcal{H}$ is complete if and only if

i) for every $x \in \mathcal{H}$

$$x = \sum_{k=1}^{\infty} \langle \phi_k, x \rangle \phi_k$$

ii) for every $x \in \mathcal{H}$

$$\|x\|^2 = \sum_{k=1}^{\infty} |\langle \phi_k, x \rangle|^2$$

iii) For every pair $x, y \in \mathcal{H}$

$$\langle x, y \rangle = \sum_{k=1}^{\infty} \langle x, \phi_k \rangle \langle \phi_k, y \rangle$$

iv) $\mathcal{H} = M[\phi_1, \phi_2, \dots]$

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where $M[\phi_1, \phi_2, \dots]$ represents the closure of the linear manifold spanned by the orthonormal system

Comment: Not all infinite dimensional linear manifolds in H are closed. The textbook (following Kolmogoroff and others) uses subspace for closed linear manifolds.

Convergence in Hilbert spaces:

Def: If, for a sequence $\{x_n \in H\}$ $\langle x_n, y \rangle \rightarrow \langle x, y \rangle$ for every $y \in H$, we will call $\{x_n\}$ weakly convergent to x , or $x_n \rightarrow x$ as $n \rightarrow \infty$.

Note: If $\phi_1, \phi_2, \dots$ is an orthonormal system $\sum_{k=1}^{\infty} |\langle \phi_k, x \rangle|^2 \leq \|x\|^2 \Rightarrow \langle \phi_k, x \rangle \rightarrow 0$ as $k \rightarrow \infty$. So $\{\phi_k\}$ weakly converges to θ .

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This is despite $\|\phi_k\| = 1$ for each k .

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