Classification of Singular Point

$L[u]=0$ with $L=\frac{d^2}{dz^2}+p(z)\frac{d}{dz}+q(z)$

Sturm-Liouville equation.

If p, q are [redacted] analytic [redacted] around $z=z_0$, it is regular (ordinary) point. One can Taylor expand u around $z=z_0$ and find a solution

The singular point $z=z_0$ is called a regular singular point if $(z-z_0)p(z)$ and $(z-z_0)^2 q(z)$ are [redacted] restrictions of analytic functions P, Q in the neighborhood of z_0.
Otherwise, the singular pt is an irregular singular pt. *

Let $P(z)=\sum_{n=0}^{\infty} p_n(z-z_0)^n = (z-z_0)p(z)$

$$Q(z)=\sum_{n=0}^{\infty} q_n(z-z_0)^n = (z-z_0)^2 q(z)$$

Try a solution of the form $u(z)=(z-z_0)^\alpha\left[1+\sum_{n=1} a_n(z-z_0)^n\right]$

$a_0=1$

[redacted] This is the Frobenius method of series solution.

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* $u''+\frac{B}{z^2}u'+\frac{2B}{z^3}u=0$ has a soln

Like $u(z)=Ae^{B/z}$

Essential singularity!

$(u')'=\left(-\frac{AB}{z^2}e^{B/z}\right)'=\frac{AB^2}{z^4}e^{B/z}+\frac{2AB}{z^3}e^{B/z}=\left[\frac{B^2}{z^4}+\frac{2B}{z^3}\right]u$

$u'=-\frac{AB}{z^2}e^{B/z}=-\frac{B}{z^2}u$

PROJECT

$$u''(z) = \sum_{n=0}^{\infty} (\alpha+n)(\alpha+n-1)\, a_n (z-z_0)^{n+\alpha-2}$$

$$p(z)\,u'(z) = \sum_{m=0}^{\infty} p_m (z-z_0)^{m-1} \sum_{k=0}^{\infty} (\alpha+k)\, a_k (z-z_0)^{k+\alpha-1}$$

$$q(z)\,u(z) = \sum_{m=0}^{\infty} q_m (z-z_0)^{m-2} \sum_{k=0}^{\infty} a_k (z-z_0)^{k+\alpha}$$

$$\Rightarrow \quad (\alpha+n)(\alpha+n-1)a_n + \text{[redacted]} \sum_{k=0}^{n} (\alpha+k) p_{n-k} a_k + \sum_{k=0}^{n} q_{n-k} a_k \text{ [redacted]} = 0$$

$n=0$ gives

$$\alpha(\alpha-1)a_0 + \alpha p_0 a_0 + q_0 a_0 = 0$$

So $\alpha(\alpha-1) + p_0\alpha + q_0 = 0$ Indicial Eqn

This is a quadratic equation for α with two roots $\alpha_\pm = \dfrac{1-p_0 \pm \sqrt{(1-p_0)^2 - 4q_0}}{2}$

Starting with an α root one can obtain the other coefficients

$[\alpha(\alpha+1) + (\alpha+1)p_0 + q_0]\, a_1 + \alpha p_1 + q_1 = 0$ Determines a_1

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$[(\alpha+n)(\alpha+n-1) + (\alpha+n)p_0 + q_0]\, a_n + [(\alpha+n-1)p_1 + q_1]\, a_{n-1} + \cdots\cdots + [(\alpha+1)p_{n-1} + q_{n-1}]\, a_1 + \alpha p_n + q_n = 0$ Determines a_n in terms of $a_1 \ldots a_{n-1}$

unless $(\alpha+n)(\alpha+n-1) + (\alpha+n)p_0 + q_0 = 0$ for some n. When we can determine these [redacted] coeffs we get convergent series (not showing that).

This observation leads[redacted] to the different possible cases we have to deal with

Call $\alpha_\pm = \frac{1}{2}\left(1 - p_0 \pm \sqrt{(1-p_0)^2 - 4q_0}\right)$

[redacted] $\alpha_+ - \alpha_- = n$, a non negative integer, we have a problem. In that case, the "series" [redacted] for α_- cannot be continued, since
$$(\alpha_-+n)(\alpha_-+n-1) + (\alpha_-+n)p_0 + q_0 = \alpha_+(\alpha_+-1) + \alpha_+ p_0 + q_0 = 0$$

So there are three cases

a) $\alpha_+ - \alpha_-$ not an integer:

Then we get two independent solutions

$$u_\pm(z) = (z-z_0)^{\alpha_\pm}\left[1 + \sum_{n=1}^{\infty} a_n^{(\pm)}(z-z_0)^n\right] = (z-z_0)^{\alpha_\pm} f_\pm(z)$$

b) $\alpha_+ = \alpha_- = \alpha$:

We get only one series $u(z) = (z-z_0)^\alpha\left[1 + \sum_{n=1}^{\infty} a_n (z-z_0)^n\right]$
$$= (z-z_0)^\alpha f(z)$$

c) $\alpha_+ - \alpha_- = n$, a positive integer:

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PROJECT

We only get one solution from the method, namely

$$u_+(z) = (z-z_0)^{\alpha_+}\left[1+\sum_{n=1}^{\infty} a_n^{(+)}(z-z_0)^n\right] = (z-z_0)^{\alpha_+} f_+(z)$$

$f(z), f_\pm(z)$ have value 1 at $z = z_0$ and are analytic in the neighborhood of z_0.

With second order differential equations, we expect two independent solutions.
What to do in case b) and c)?

Here, the Wronskian comes to rescue
If we have only one solution, $u_+(z)$, and we are try to find another independent solution, $u(z)$

$$W(z) = \begin{vmatrix} u_+(z) & u(z) \\ u_+'(z) & u'(z) \end{vmatrix}$$ must be non zero somewhere.

We also know that $W(z) = C \exp\left\{-\int^z p(\xi)\,d\xi\right\}$

$$W(z) = u_+(z)\,u'(z) - u_+'(z)\,u(z) = \frac{u_+(z)u'(z) - u_+'(z)u(z)}{u_+(z)^2}\,u_+(z)^2$$

$$= \left(u_+(z)\right)^2 \frac{d}{dz}\left(\frac{u(z)}{u_+(z)}\right)$$

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So $\frac{d}{dz}\left(\frac{u(z)}{u_+(z)}\right) = \frac{C}{[u_+(z)]^2} e^{-\int^z p(\xi)d\xi}$

If $p(z) = \frac{p_0}{z-z_0} + r(z) = \frac{1 \cancel{-\alpha_+} - \alpha_-}{z - z_0} + r(z) = \frac{n+1-2\alpha_+}{z-z_0} + r(z)$

$r(z)$ analytic

$$e^{-\int^z r(\xi)d\xi} = \exp\left[-(n+1-2\alpha_+)\ln(z-z_0) - \int^z r(\xi)d\xi\right]$$

$$= \frac{(z-z_0)^{2\alpha_+}}{(z-z_0)^{n+1}} e^{-\int^z r(\xi)d\xi}$$

Since $u_+(z) = (z-z_0)^{\alpha_+} f_+(z)$

$$\frac{d}{dz}\left(\frac{u(z)}{u_+(z)}\right) = \frac{C}{(z-z_0)^{n+1}} \frac{e^{-\int^z r(\xi)d\xi}}{[f_+(z)]^2} = \frac{K}{(z-z_0)^{n+1}} g(z)$$

$g(z)$ analytic around $z=z_0$

Let $g(z) = \sum_{m=0}^{\infty} \frac{g^{(m)}(z_0)}{m!}(z-z_0)^m$

$$\frac{u(z)}{u_+(z)} = K_1 + K\int^z \frac{g(\xi)}{(\xi-z_0)^{n+1}} d\xi$$

$$= K_1 + K\left[\sum_{\substack{m=0\\ m\neq n}}^{\infty} \frac{g^{(m)}(z_0)}{m-n} \frac{(z-z_0)^{m-n}}{n!} + \frac{g^{(n)}(z_0)}{n!}\ln(z-z_0)\right]$$

So the other soln is $u_-(z) = u_+(z)\left[\sum_{\substack{m=0\\ m\neq n}}^{\infty} \frac{g^{(m)}(z_0)}{m-n} \frac{(z-z_0)^{m-n}}{n!} + \frac{g^{(n)}(z_0)}{n!}\ln(z-z_0)\right]$

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Using $u_+(z) = (z-z_0)^{\alpha_+} f_+(z) = (z-z_0)^{\alpha_- + n} f_+(z)$

$$u_-(z) = (z-z_0)^{\alpha_-} f_-(z) + \frac{g^{(n)}(z_0)}{n!} u_+(z)\ln(z-z_0)$$

where $$f_-(z) = f_+(z) \sum_{\substack{m=0 \\ m \neq n}}^{\infty} \frac{g^{(m)}(z_0)}{m-n} \frac{(z-z_0)^m}{m!}$$

$f_-(z)$ is analytic around $z = z_0$

The key point is that $u_-(z)$ has a logarithmic singularity from the term [redacted] $\frac{g^n(z_0)}{n!} \ln(z-z_0)\, u_+(z)$, that behaves like $(z-z_0)^{\alpha_+} \ln(z-z_0)$ near the singularity, (in addition to possible singularity of the form $(z-z_0)^{\alpha_-} = (z-z_0)^{\alpha_+ - n}$).

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Legendre's Equation

Legendre Polynomial

$$(z^2-1)\,u''(z) + 2z\,u'(z) - \lambda(\lambda+1)\,u(z) = 0$$

[Came from $\frac{d}{dz}(1-z^2)\frac{d}{dz}\,u(z) = -\lambda(\lambda+1)\,u(z)$

with $z=\cos\theta$ transformation. Originally $\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta}f = -\lambda(\lambda+1)f$]

$$u''(z) + \underbrace{\frac{2z}{z^2-1}}_{p(z)}\,u'(z) - \underbrace{\frac{\lambda(\lambda+1)}{z^2-1}}_{q(z)}\,u(z) = 0$$

Reg. Singularities at $z=\pm 1$

Let us try to expand around $z=0$, a regular point,

$$u(z) = \sum_{k=0}^{\infty} a_k z^k$$

$$(z^2-1)\sum_k \overset{k(k-1)}{a_k}\, z^{k-2} + 2z\sum_k k a_k z^{k-1} - \lambda(\lambda+1)\sum a_k z^k = 0$$

[redacted] $\sum_k k(k-1)a_k z^k - \sum_k k(k-1)a_k z^{k-2} + \sum_k 2k a_k z^k - \sum_k \lambda(\lambda+1)\,a_k z^k = 0$

$$(k+1)(k+2)\,a_{k+2} = [k(k+1)-\lambda(\lambda+1)]\,a_k$$

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So $$\frac{a_{k+2}}{a_k} = \frac{k(k+1)-\lambda(\lambda+1)}{(k+1)(k+2)} = \frac{(k-\lambda)(k+\lambda+1)}{(k+1)(k+2)}$$

As $k \to \infty$, this ratio [redacted] tend to 1, telling us that the radius of convergence of the series is 1. This, we expected from the singularities [redacted] of $p(z)$, $q(z)$.

However, if λ is a nonnegative integer n, this series would terminate and we get a polynomial, an entire function.

$P_n(z)$ are the Legendre polynomials

$P_0(z) = 1$, $P_1(z) = z$, $P_2(z) = \frac{1}{2}(3z^2 - 1)$, ...

with the normalization $P_n(1) = 1$

What about general λ? Let us try to expand around $z = 1$ and use variable t s.t.

$z = 1 - 2t$

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$$(z^2-1)\frac{d^2}{dz^2} + 2z\frac{d}{dz} - \lambda(\lambda+1)$$

$$= \left[(1-2t)^2-1\right]\frac{1}{4}\frac{d^2}{dt^2} + 2(1-2t)\left(-\frac{1}{2}\right)\frac{d}{dt} - \lambda(\lambda+1)$$

$$= t(t-1)\frac{d^2}{dt^2} + (2t-1)\frac{d}{dt} - \lambda(\lambda+1) \quad \Rightarrow \quad \boxed{t(t-1)u''(t) + (2t-1)u'(t) - \lambda(\lambda+1)u(t) = 0}$$

[redacted]

$$p(t) = \frac{2t-1}{t(t-1)}, \qquad q(t) = -\frac{\lambda(\lambda+1)}{t(t+1)}$$

$$p(t) \sim +\frac{1}{t}, \qquad q(t) \sim \frac{\lambda(\lambda+1)}{t}$$

$$\Rightarrow p_0 = +1 \qquad \Rightarrow q_0 = 0$$

Indicial eqn

$$\alpha(\alpha-1) + \alpha p_0 + q_0 = 0 \quad \Rightarrow \quad \alpha(\alpha-1) + \alpha = 0$$

So $\alpha^2 = 0$

So we have soln [redacted] $P_\lambda(z) = u\left(\frac{1-z}{2}\right)$

$$u(t) = 1 + \sum_{k=1}^{\infty} c_k t^k \qquad c_0 = 1$$

$$k(k-1)c_k \text{ [redacted] } -(k+1)k\,c_{k+1} + 2k\,c_k - (k+1)c_{k+1} - \lambda(\lambda+1)c_k = 0$$

$$(k+1)^2 c_{k+1} = \left[k(k+1) - \lambda(\lambda+1)\right]c_k = (k-\lambda)(k+\lambda+1)c_k$$

$$c_k = \frac{\Gamma(k-\lambda)\Gamma(k+\lambda+1)}{\Gamma(-\lambda)\Gamma(\lambda+1)}\frac{1}{(k!)^2}$$

$P_\lambda(z)$ is the Legendre function of the first kind.

There ought to be a second solution $Q_\lambda(z)$ with logarithmic [redacted] singularity around $z = \pm 1$.
This is the Legendre function of the second kind.

Alternatively one can look at behavior for $|z| > 1$.
One way to do that is use varible $\xi = \frac{1}{z}$.
The Legendre eqn now becomes

$$\left[\xi^2 \frac{d}{d\xi}(\xi^2 - 1)\frac{d}{d\xi} + \lambda(\lambda+1)\right] u(1/\xi) = 0$$

$$\Rightarrow \left(\frac{d^2}{d\xi^2} + \frac{2\xi}{\text{[redacted]}(\xi^2-1)}\frac{d}{d\xi} + \frac{\lambda(\lambda+1)}{\xi^2(\xi^2-1)}\right) u(1/\xi) = 0$$

The differential operator in ξ, in the final version has coefficients $p(\xi) = \frac{2\xi}{\text{[redacted]}(\xi^2-1)}$, $q(\xi) = \frac{\lambda(\lambda+1)}{\xi^2(\xi^2-1)}$

So it has regular singular points at $\xi = 0, \pm 1$

[redacted] Q_λ is the series solution around $\xi = 0$.

$Q_\lambda(\frac{1}{\xi}) \sim \xi^\alpha(1 + \xi^2 + \cdots)$

α satisfies $\alpha(\alpha - 1) =$ [redacted] $\lambda(\lambda+1) \Rightarrow \alpha = \lambda+1, -\lambda$

$\alpha = \lambda + 1$ solution gives Q_λ.

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Schläfli Integral representation

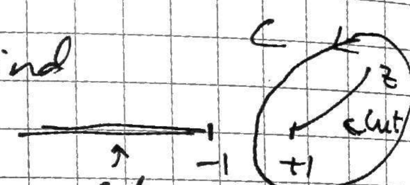
$$I_\lambda(z) = K_\lambda \int_C \frac{(1-\xi^2)^\lambda}{(z-\xi)^{\lambda+1}}\, d\xi$$

By showing that $\left[\frac{d}{dz}(z^2-1)\frac{d}{dz} - \lambda(\lambda+1)\right]\frac{(1-\xi^2)^\lambda}{(z-\xi)^{\lambda+1}}$

$$= -\frac{d}{d\xi}\frac{(1-\xi^2)^{\lambda+1}}{(z-\xi)^{\lambda+2}}$$

namely a total derivative in ξ, we guarantee that $I_\lambda(z)$ would satisfy the Legendre equation, because the closed contour integral of the total derivative would vanish. Different choices of C correspond to different solutions.

Legendre function of the first kind



See Vaughn page 233 for $\nu = 0, 1, 2, \ldots$

For comment on Schläfli representation of $Q_\nu(z)$, see Vaughn, page 236.

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Bessel's Equation

Bessel Function

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} + \left(1 - \frac{\lambda^2}{z^2}\right)u = 0$$

[Came from $\frac{1}{z}\frac{d}{dz} z \frac{d}{dz} u - \frac{\lambda^2}{z^2} u = -u$. In cylindrical coordinates
$\left(\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{d}{d\rho} - \frac{m^2}{\rho^2}\right)f = -k^2 f$]

We have a regular singular point at $z=0$

$p(z) = \frac{1}{z}$, $q(z) = 1 - \frac{\lambda^2}{z^2}$ $\Rightarrow$ $p_0 = 1$, $q_0 = -\lambda^2$

Indicial eqn. $\alpha(\alpha-1) + \alpha - \lambda^2 = 0$

$\Rightarrow \alpha^2 = \lambda^2 \Rightarrow \alpha = \pm\lambda$

Choose $\alpha = \lambda$, $u(z) = z^\lambda\left(1 + \sum_{k=1}^{\infty} a_k z^k\right)$, $a_0 = 1$

$$\left[(k+\lambda+2)^2 - \lambda^2\right]a_{k+2} + a_k = 0$$

If we start with $a_0 = 1$ we only have even powers

$$a_{2m} = -\frac{1}{(2m+\lambda)^2 - \lambda^2} a_{2m-2} = -\frac{1}{4m(m+\lambda)} a_{2m-2}$$

$$J_\lambda(z) = \left(\frac{z}{2}\right)^\lambda \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\lambda+1)}\left(\frac{z}{2}\right)^{m}$$

This is the Bessel function of order λ.

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To get the other solution for nonintegers λ, one considers $\alpha = -\lambda$

Conventionally, one defines

$$N_\lambda(z) = \frac{J_\lambda(z)\cos\pi\lambda - J_{-\lambda}(z)}{\sin\pi\lambda}$$

which is called Bessel func. of 2nd kind, also called Neumann func. This [redacted] combination has a well defined limit when λ goes to an integer value. The limit, as expected, has logarithms showing up in expansion around $z=0$.

Bessel functions also have integral representation. For integer order,

$$J_n(z) = \frac{1}{2\pi i}\oint_C e^{\frac{z}{2}\left(t-\frac{1}{t}\right)}\frac{dt}{t^{n+1}}$$

These relations are useful for asymptotics

Legendre equation and Bessel equation are related to a broader class of differential equation. Equations with only three regular singular points (like ±1, ∞ for Legendre eqn) in the Riemann sphere could be tranformed into the hypergeometric equation. Legendre equation, therefore, is related to a special case of hypergeometric function.

A confluence of two singular points lead to the confluent hypergeometric equation. Bessel equation, with singularities only at $z=0$ and $z=\infty$ is related to a special case of this category.

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