Def: A series [redacted] corresponding to a sequence $\{z_k\}$ is the sequence of partial sums $\{S_n\}$ where

$$S_n = \sum_{k=1}^{n} z_k .$$

The series is often written as $\sum_{k=1}^{\infty} z_k$

$\sum_k z_k$ is convergent if $\{S_n\}$ has a limit.

~~This theorem concerns real sequences.~~
~~Def: A sequence that is either increasing~~ [redacted] ~~or decreasing is~~
~~Theorem: A monotone sequence is convergent~~
~~if and only if it is bounded.~~
~~Proof: Exercise~~

Def: [redacted] A series $\sum_k z_k$ is absolutely convergent if $\sum_k |z_k|$ converges

A [redacted] convergent series that is not absolutely convergent is called conditionally convergent.

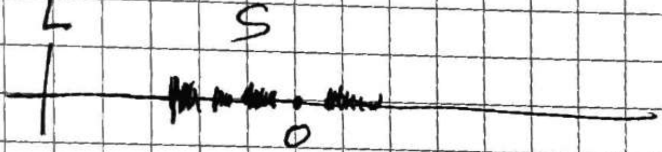
Continued on Page

We will now launch into a discussion of monotonic sequences and Cauchy sequences.

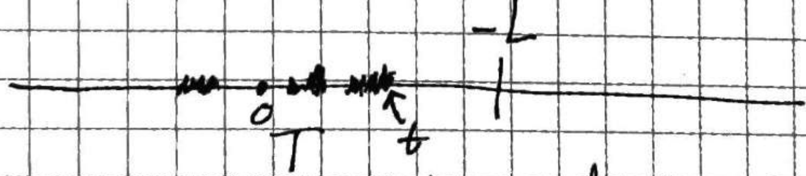
Before we go there, note that the least upper bound property implies the greatest lower bound property. The greatest lower bound of S is represented as $\inf S$.

Proposition: If a real set S is nonempty and bounded below, it has a greatest lower bound.

Proof:

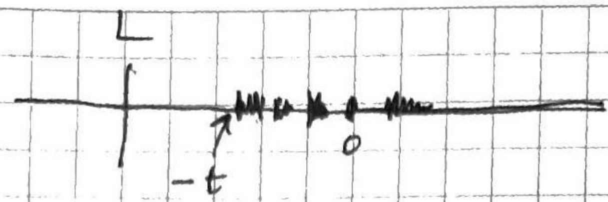


Construct $T = \{-x \mid x \in S\}$. It is nonempty and has an upper bound $-L$.



Thus, we have a least upper bound $t = \sup T$.
Now, $-t$ is going to be the greatest upper bound of S.

Continued on Page



Why? If the was a lower bound l, s.t. $-t < l$, but $x \geq l$ for all $x \in S$, then $-l \geq -x$ $\forall x \in S \Rightarrow -l$ is an upper bound of S. Also $-t < l \Rightarrow t > -l \Rightarrow t$ is not the least upper bound. Contradiction!

QED

Now let us come to the first theorem concerning real sequences.

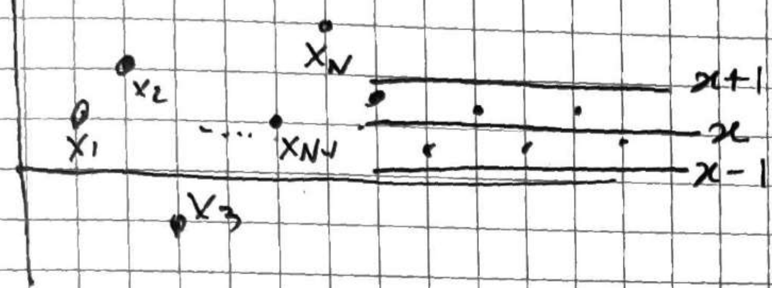
Thm: Every monotonic sequence is convergent if and only if it is bounded.

The 'only if' part is easy, since any convergent sequence is bounded. Let $\{x_n\}$ have a limit x. Then there is an N, s.t.:

Continued on Page

$n > N \Rightarrow |x_n - x| < 1$ (I chose $\varepsilon = 1$ in our definition of convergent sequence.)

So $x-1 < x_n < x+1$ for $n > N$

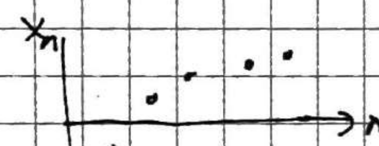


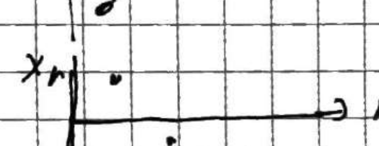
So the sequence is bounded above by $\max\{x_1, \dots, x_N, x+1\}$ and bounded below by $\min\{x_1, \dots, x_N, x-1\}$.

Now, the 'if' part:

Case a) $\{x_n\}$ is increasing

b) $\{x_n\}$ is decreasing



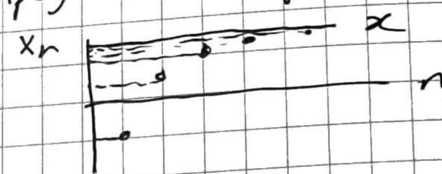


We will prove case a) using the least upper bound property. Case b) can be done analogously, by using the greatest lower bound property.

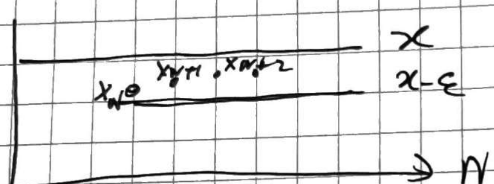
Continued on Page

Consider the set $S = \{x_n \mid n \in \mathbb{N}\}$

If S is bounded, it has an upper bound. It is obviously non-empty. So $\sup S$ exists. Set $x = \sup S$.



For any $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $x_N > x - \varepsilon$.
(otherwise, x is not the least upper bound).



For $n > N$
$x_n \geq x_N > x - \varepsilon$
Also $x_n \leq x$

For $n > N$, therefore, $|x_n - x| < \varepsilon$.

So $\{x_n\}$ converges to x.

Flip things around for decreasing sequence.
Show that $\inf\{x_n \mid n \in \mathbb{N}\}$ is the limit in that case. QED.

Continued on Page

Let us pick an application. Consider the sequence $\left\{\left(1+\frac{1}{n}\right)^n\right\}$. Is it convergent?

I am sure all of you know that this sequence converges to e. Sometimes it is used to define e.

Let us see if we can prove its convergence.

$x_1 = 2$, $x_2 = (1+\frac{1}{2})^2 = \frac{9}{4} = 2.25$, Ok, maybe it is increasing!

$$x_n = \left(1+\frac{1}{n}\right)^n = 1 + n\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \cdots + \frac{n(n-1)\cdots(n-k+1)}{k!}\frac{1}{n^k} + \cdots + \frac{n(n-1)\cdots 1}{n!}\frac{1}{n^n}$$

$$= 1 + 1 + \frac{1}{2!}\,1\cdot\left(1-\frac{1}{n}\right) + \cdots + \frac{1}{k!}\,1\cdot\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) + \cdots + \frac{1}{n!}\,1\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$$

Compare with $\left(1+\frac{1}{n+1}\right)^{n+1}$

$$x_{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1} = 1 + 1 + \frac{1}{2}\,1\left(1-\frac{1}{n+1}\right) + \cdots + \frac{1}{k!}\,1\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{k-1}{n}\right) + \cdots + \frac{1}{n!}\,1\cdot\left(1-\frac{1}{n+1}\right)\cdots\left(1-\frac{n-1}{n+1}\right)$$
$$+ \frac{1}{(n+1)!}\,1\left(1-\frac{1}{n+1}\right)\cdots\left(1-\frac{n}{n+1}\right)$$

The kth term has gotten bigger since $\left(1-\frac{l}{n}\right) < \left(1-\frac{l}{n+1}\right)$ for $l \le n$. On top of that x_{n+1} has an additional positive term.

Continued on Page

Continued From Page

So $x_{n+1} > x_n$

Also $x_n = 1+1+\frac{1}{2!}1\cdot(1-\frac{1}{n}) + \cdots + \frac{1}{k!}1(1-\frac{1}{n})\cdots(1-\frac{k-1}{n})$

$+ \cdots + \frac{1}{n!}1(1-\frac{1}{n})\cdots(1-\frac{k-1}{n})$

$< 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{k!}+\cdots+\frac{1}{n!}$

Could I use it to prove $\{x_n\}$ is bounded above?

We will reuse it else where, but

$1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}$ is bounded above and could be used to prove that the series $\sum_{k=1}^{\infty}\frac{1}{k!}$ (another definition of e) is also convergent.

$1+1+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\frac{1}{1\cdot 2\cdot 3\cdot 4}+\cdots+\frac{1}{1\cdot 2\cdot 3\cdots n}$

$< 1+1+\frac{1}{2}+\frac{1}{2\cdot 2}+\frac{1}{2\cdot 2\cdot 2}+\cdots+\frac{1}{2^{n-1}}$

Continued on Page

$$= 1 + \frac{1-\frac{1}{2^n}}{1-\frac{1}{2}} = 1 + 2\left(1-\frac{1}{2^n}\right) < 1+2 = 3$$

So $x_n < 3$ for all n. Also $x_n \geq 2$

Since $\{x_n\}$ is monotonic and bounded, it must have a limit.

Note that we also ended up proving $\sum_{k=1}^{\infty} \frac{1}{k!}$ is convergent, since partial sums keep on increasing and the sum is always bounded above.

Our proof already shows $\lim_{n\to} (1+\frac{1}{n})^n \leq \sum_{k=1}^{\infty} \frac{1}{k!}$

What about the other way around? Turns out that $\sum_{k=1}^{\infty} \frac{1}{k!} \leq \lim_{n\to\infty} (1+\frac{1}{n})^n$

$$x_n = 1 + 1 + 1(1-\tfrac{1}{n})\tfrac{1}{2!} + \cdots + 1(1-\tfrac{1}{n})\cdots(1-\tfrac{k-1}{n})\tfrac{1}{k!} + \cdots + 1(1-\tfrac{1}{n})(1-\tfrac{2}{n})\cdots\tfrac{1}{n!}$$

Stop at a fixed k.

For $n \geq k$ $\quad x_n \geq 1 + 1 + 1(1-\frac{1}{n})\frac{1}{2!} + \cdots + 1\cdot(1-\frac{1}{n})(1-\frac{k-1}{n})\frac{1}{k!}$

By taking $n \to \infty$, $x \geq 1 + 1 + \cdots + \frac{1}{k!}$ (This relies on

Continued on Page

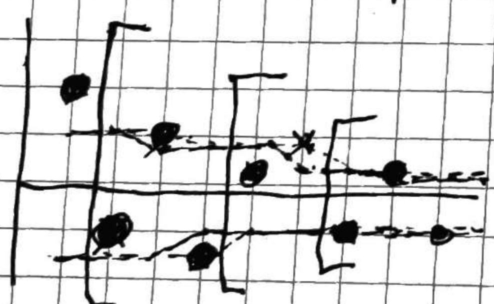
theorems on limits of sums and products, which I am assuming that you know. [redacted] We also need that if two increasing convergent sequences $\{a_n\}, \{b_n\}$ satisfy $a_n \leq b_n \ \forall n \in \mathbb{N}$, then $\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n$

The last bit could have been helped by the concepts of lim inf and lim sup. They also help us prove [redacted] statements about real Cauchy sequences. [redacted] They would also be useful for statements of root test and ratio test.

Def: Let $\{x_n\}$ [redacted] be a real sequence

Define $u_n = \sup\{x_m \mid m > n\}$

$l_n = \inf\{x_n \mid m > n\}$



Note that $\{u_n\}$ is decreasing while $\{l_n\}$ is increasing

Continued on Page

For monotonic sequences, We know the bounded ones converge. If they are not bounded, increasing ones go to $+\infty$ and decreasing ones go to $-\infty$. If we allow $\pm\infty$ as possible limits limit of a monotonic sequence is always defined.

Finally we define lim sup & lim inf

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} u_n$$

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} l_n$$

By construction $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$

For convergent sequences, These two are the same!