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Differential Equation : Analytic Methods

In this section we will take a look at differential equations, takin advantage if possible of things we learnt about (complex) analytic functions.

General system of first order equations

$$\frac{d}{dz} u_k(z) = h_k(u_1, \ldots, u_n; z) \qquad k = 1, \ldots, n$$

Note that a general n-th order diff. eq. can be recast this way, at least locally.

$$F(v, v', \ldots, v^{(n)}, z) = 0$$

Solve this to rewrite

$$v^{(n)} = f(v, v', \ldots, v^{(n-1)}; z)$$

Now define $u_k = v^{(k-1)}$, namely $u_1 = v, u_2 = v', \ldots, u_n = v^{(n-1)}$

$$u_1' = u_2$$
$$u_2' = u_3$$
$$u_{n-1}' = u_n$$
$$u_n' = f(u_1, \ldots, u_n; z)$$

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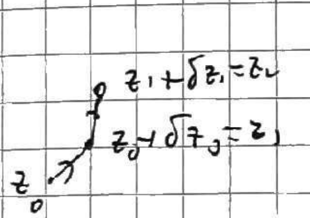
Example 1: [redacted] $m\frac{d^2x}{dt^2} = -V'(x)$ } One second-order equation

$$u_1 = x \qquad u_2 = \frac{dx}{dt}$$

$$\left.\begin{aligned}\frac{du_1}{dt} &= u_2\\ \frac{du_2}{dt} &= -\frac{1}{m}V'(u_1)\end{aligned}\right\}$$ Two first-order equations.

For the general first-order equation we need $u_k(z_0) = u_{k0}, k=1,n$ the 'initial conditions' to 'determine' a solution. Intuition!



$$\overset{\underline{u}(z_1)}{=} \underline{u}(z_0+\delta z_0) \approx \underline{u}(z_0) + \underline{h}(z_0)\,\delta z_0$$ [redacted]

$$\underline{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \qquad \underline{h} = \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}$$

$$\underline{u}(z_1+\delta z_1) = \underline{u}(z_1) + \underline{h}(z_1)\,\delta z_1 \qquad \text{etc etc.}$$

Can this intuition be made to work analytically or numerically?

For complex analytic diff eqs solution may be path dependent [redacted] and depend on analytic continuation.

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For an autonomous system

$$\frac{du_k}{dt} = h_k(u_1, \dots, u_n) \qquad k = 1, \dots, n$$

One can make it locally into an $n-1$ variable non autonomous system.

Call $u_n = \tau$

$$\frac{du_k}{d\tau} = \frac{h_k(u_1, \dots, u_{n-1}, \tau)}{h_n(u_1, \dots, u_{n-1}, \tau)} \qquad k = 1, \dots, n-1$$

Example: $m\ddot{x} = -kx$

$$\Rightarrow \left.\begin{aligned} \frac{du_1}{dt} &= u_2 \\ \frac{du_2}{dt} &= -\frac{k}{m} u_1 \end{aligned}\right\} \quad \begin{aligned} u_1 &= x \\ u_2 &= \dot{x} \end{aligned}$$

Note that $\frac{du_1}{du_2} = -\frac{m}{k}\frac{u_2}{u_1}$

Use variabler separable method $\int u_1 du_1 = -\frac{m}{k}\int u_2 du_2$

$\Rightarrow$ [redacted] $\frac{m u_2^2}{2} + \frac{k u_1^2}{2} = const$

$$\left[\frac{m}{2}\left(\frac{dx}{dt}\right)^2 + \frac{k}{2}x^2 = Energy \right]$$

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Scale invariant equations:

$z \to az$ an invariance

Example: $\frac{du}{dz} =$ [redacted] $\frac{1}{z} g(u)$

Trick: $z\frac{du}{dz} = g(u) \Rightarrow \frac{du}{d\tau} = g(u)$ where $\tau = \ln z$

$\Rightarrow \int \frac{du}{g(u)} = \tau + cst. \Rightarrow z = z_0 e^{\int_{u_0}^{u} \frac{dv}{g(v)}}$

For $g(u) = \gamma u$ $\quad z = c_0 e^{\frac{1}{\gamma} u} \Rightarrow \left(\frac{z}{z_0}\right) = \left(\frac{u}{u_0}\right)^{\frac{1}{\gamma}}$

$\Rightarrow u = u_0 \left(\frac{z}{z_0}\right)^{\gamma}$

Isobaric ODE: $z \to az \quad u \to a^p u$
(scale covariant)

is an invariance

Try $u = z^p v$. The fnc v is invariant $\Rightarrow$ scale invariant equation.

Example: $\frac{du}{dz} = \frac{u}{z} - z$

$u \to a^2 u \quad z \to az$ is an invariance

Introduce v s.t. $u = z^2 v$

Note v invariant under transformation

$z^2 \frac{dv}{dz} + 2zv = zv - z$ } Get rid of extra factor of z

$z\frac{dv}{dz} = -v - 1$ } Now, ready to use tricks from scale inv equation

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$\ln z = -\ln(v+1) + const$

$v =$ [redacted] $\frac{cst - z}{z} \Rightarrow u = z(A - z)$

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First Order Diff Eqs

Linear first order eqs

Homogeneous Linear Case

$u'(z) + p(z)\,u(z) = 0$

$$\frac{du}{u} = -p \implies u(z) = u(z_0)\, e^{-\int_{z_0}^{z} p(\xi)\,d\xi}$$

BTW, notice that if $p(z)$ has a simple pole at z_1, $\underbrace{}_{p(\xi)\simeq\frac{\alpha}{\xi-z_1}+\cdots}$ then $\int^{z} p(\xi)\,d\xi \sim \alpha \ln(z-z_1) + \cdots$

and $u(z) \sim (z-z_1)^{\alpha}$. Branch pt if $\alpha \notin \mathbb{Z}$

So meromorphic "coefficients" could give rise to solutions with branch points

Now we can solve the inhomogeneous problem

$$u'(z) + p(z)\,u(z) = f(z)$$

Define $h(z) = e^{-\int_{z_0}^{z} p(\xi)\,d\xi}$ (Soln of the homogeneous problem)

and let $u(z) = h(z)\,w(z)$

$$h(z)\,w'(z) = f(z) \implies w(z) = u(z_0) + \int_{z_0}^{z} \frac{f(\xi)}{h(\xi)}\,d\xi$$

$$\implies u(z) = \left(u(z_0) + \int_{z_0}^{z} \frac{f(\xi)}{h(\xi)}\,d\xi\right) h(z)$$

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This is just the method of integrating factor.

Ricatti equation

We will mention a particular kind of first order nonlinear differential equation, namely Ricatti eqn.

$u' = q_0(z) + q_1(z)u + q_2(z)u^2$

It [redacted] is interesting that this eqn can be transformed into a linear second order equation.

We have seen such a thing in the context of quantum mechanics

$\psi \sim e^{iS/\hbar}$ $\qquad -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$

$\Rightarrow -\frac{\hbar^2}{2m}\left[\left(\frac{i}{\hbar}S'\right)^2 + \frac{i}{\hbar}S''\right] + V(x) - E = 0$

If [redacted] $\frac{S'}{\hbar} = u$ [redacted] $u' = \frac{2mi}{\hbar}(E - V(x))$ [redacted] $-\frac{i}{2m}u^2$

This the beginning of the [redacted] WKB approximation

We thus try $y(z) = e^{-\int^z q_2(z')u(z')dz'}$

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$$\frac{y'(z)}{y(z)} = \frac{d}{dz}\ln y(z) = -q_2(z)\,u(z)$$

$$u(z) = -\frac{1}{q_2(z)}\frac{y'(z)}{y(z)}$$

$$u'(z) = -\frac{q_2'(z)}{q_2(z)^2}\frac{y'(z)}{y(z)} + \frac{1}{q_2(z)}\left(\frac{y'(z)}{y(z)}\right)^2 - \frac{1}{q_2(z)}\frac{y''(z)}{y(z)}$$

$q_0(z)+q_1(z)u(z)+q_2(z)u(z)^2 = q_0(z)$ [redacted] $- \frac{q_1(z)}{q_2(z)}\frac{y'(z)}{y(z)} + \frac{q_2(z)}{q_2(z)^2}\left(\frac{y'(z)}{y(z)}\right)^2$

So Ricatti eqn $\Rightarrow$ $y''(z) - \left(\frac{q_2'(z)}{q_2(z)} + q_1(z)\right)y'(z) + q_0(z)q_2(z)y(z) = 0$

which is a second order linear differential equation. We will discuss such equations soon.

Interestingly, if we know one particular solution of the Ricatti equation of the form $u_0(z)$, we can try to find a class of solutions as [redacted] follows:

Let $u(z) = u_0(z) - \frac{c}{v(z)}$, while $u_0'(z) = q_0(z) + q_1(z)u_0(z) + q_2(z)u_0(z)^2$

$$u'(z) = u_0'(z) + \frac{c}{v(z)^2}v'(z)$$

$$q_0(z) + q_1(z)\left(u_0(z) - \frac{c}{v(z)}\right) + q_2(z)\left(u_0(z)^2 - \frac{2c\,u_0(z)}{v(z)} + \frac{c^2}{v(z)^2}\right)$$

$$= q_0(z) + q_1(z)u_0(z) + q_2 u_0(z)^2 - \frac{c}{v(z)^2}\left\{q_1(z)v(z) + 2q_2(z)u_0(z)v(z) - c\,q_0(z)\right\}$$

$$\Rightarrow \quad v'(z) + \left[q_1(z) + 2q_2(z)\,u_0(z)\right]v(z) = c\,q_2(z)$$

which can be solved by the method of integrating factor.

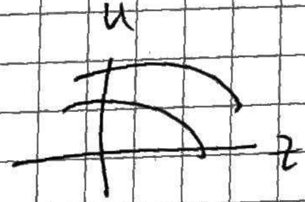
General first-order differential equation, exact differentials and integrating factors.

The general first-order equation is of the form

$$g(u,z)\,du + h(u,z)\,dz = 0$$

If $g(u,z) = \frac{\partial F}{\partial u}$ and $h(u,z) = \frac{\partial F}{\partial z}$

then the equation is just $dF = 0$



$\Rightarrow F(u,z) =$ constant is the solution.

If the oneform is dF, it is exact.

If the one form is not exact one may find $\lambda(u,z)$ so that $\lambda(u,z)\big(g(u,z)\,du + h(u,$ [redacted] $)\,dz\big)$ [redacted] is an exact form. Such a λ is called an integrating form. Its existence can be proved in many cases but finding it practically may be hard.

[what we really mean is that we have a curve $\gamma : t \longmapsto$ (u [redacted] t), $z(t)$), so that $(g\,du + h\,dz)\left(\frac{d\gamma}{dt}\right) = 0$. Such a curve [redacted] would be a parametrized solution. [redacted] The domain of t could be real intervals or complex regions depending on the context.]

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Linear [redacted] Differential Equations

Linear Systems with Constant coefficients

$$\frac{d u_k(z)}{dz} = \sum_l A_{kl}\, u_l(z) \qquad k,l = 1,\dots,n$$

$$\frac{d}{dz}\, \underline{u}(z) = \underline{A}\, \underline{u}(z)$$

$$\underline{u}(z) = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \qquad \underline{A} = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix}$$

Note that $\underline{u}(z) = e^{(z-z_0)\underline{A}}\, \underline{u}(z_0)$ solves this equation.

For the sake of convenience, Set $z_0 = 0$.

Choose a basis in which $\underline{A} \to \underline{D} + \underline{N}$

$\underline{D}$ ↑ diagonal $\quad \underline{N}$ ↑ Nilpotent

In particular choose the Jordan normal form

$$\underline{A} \to \left(\begin{array}{c|c} \begin{matrix} \lambda_1 & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & \lambda_1 \end{matrix} & \\ \hline & \lambda \end{array}\right)$$

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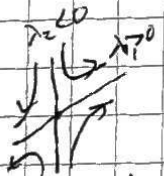
[redacted] Let $\underline{u}$[redacted]$(z) = \sum_{\ell} \xi_\ell(z)\, \underline{u}_\ell$ [redacted]

[redacted] $\underline{u}_1, \dots, \underline{u}$[redacted] being the basis vectors

Note that ξ_ℓ[redacted](z)'s corresponding to the same Jordan block affect each other.

If each block is 1 dimensional (the case where A is diagonalizable), life is simple



$$\frac{d\xi_k(z)}{d(z)} = \lambda_k \xi_k(z) \Rightarrow \xi_k(z) = e^{\lambda_k z} \xi_k(0)$$

What if there is a nilpotent component? Let us look at the 2×2 block case

$$\frac{d}{dz}\text{[redacted]} \begin{pmatrix} \xi_1(z) \\ \xi_2(z) \end{pmatrix} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \xi_1(z) \\ \xi_2(z) \end{pmatrix}$$

So $\xi_2'(z) = \lambda \xi_2(z) \Rightarrow \xi_2(z) = e^{\lambda z} \xi_2(0)$

Then $\xi_1'(z) = \lambda \xi_1(z) + \xi_2(z)$

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This is an inhomogeneous first order equation.

$$\xi_1'(z) - \lambda \xi_1(z) = \text{[redacted]}\ e^{\lambda z}\, \xi_2(0)$$

Method of integrating factor

$$\frac{d}{dz}\left(e^{-\lambda z}\, \xi_1(z)\right) = \xi_2(0)$$

Integrate from 0 to z

$$e^{-\lambda z}\, \xi_1(z) - \xi_1(0) = \xi_2(0) z \ \text{[redacted]}$$

$$\Rightarrow \quad \xi_1(z) = e^{\lambda z}\, \xi_1(0) + z e^{\lambda z}\, \xi_2(0)$$

Product of polynomials and exponentials

You can imagine, that for bigger blocks we will get higher order polynomials. If eigenvalues $\lambda_1, \dots, \lambda_q$ has multiplicities $m_1, \dots, m_q$

$u(z) = \sum_{i=1}^{q} p_i(z)\, e^{\lambda_i z}$ where $p_i(z)$ are polynomials of degree $d_i \le m_i - 1$, for $i = 1, \dots, q$.

For inhomogeneous equations

$$\frac{d}{dt} \underline{u}(z) = A \underline{u}(z) + \underline{f}(z)$$

One can see that $\underline{u}(z) = e^{Az}\, \underline{u}(0) + \int_0^z e^{A(z-\xi)}\, \underline{f}(\xi)\, d\xi$

We will use the notation L[[redacted] $\underline{u}$] $=0$, $L=\frac{d}{dz}\mathbb{1}+\underline{P}(z)$, [redacted] e.g. $L=\frac{d}{dz}\mathbb{1}-\underline{A}$

For the homogeneous equation solutions $\underline{u}_1, \underline{u}_2$ of the equation $L[\underline{u}]=0$ [redacted] lead to linear combinations $a_1\underline{u}_1+a_2\underline{u}_2$ which would also be a solution $L[u]=0$. So the solutions form a vector space.

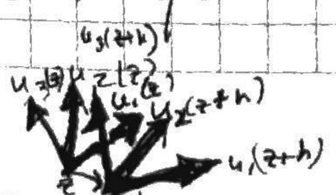
Given that the space of initial conditions are n-dimensional we expect the solution space to be n-dimensional as well.

Many of the [redacted] observations (made for [redacted] constant coefficient eq) go over to the more general homogeneous first order linear equations

$$\frac{d}{dz}\underline{u}(z)+\underline{P}(z)\,\underline{u}(z)=0$$

$\underline{P}(z)$ is a matrix of size $n\times n$ with analytic entries. The previous discussion correspond to $\underline{P}(z)=-A$, a constant matrix.

Now consider n [redacted] solutions of the linear homogeneous [redacted] equation: $\underline{u}_1$[redacted], …, $\underline{u}_n$[redacted]



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Consider the matrix $[\underline{u}_1^{\text{[redacted]}} \cdots \underline{u}_n^{\text{[redacted]}}] = \begin{bmatrix} u_{11}^{\text{[redacted]}} & u_{12}^{\text{[redacted]}} & \cdots & u_{1n}^{\text{[redacted]}} \\ \vdots & & & \vdots \\ u_{n1}^{\text{[redacted]}} & u_{n2}^{\text{[redacted]}} & \cdots & u_{nn}^{\text{[redacted]}} \end{bmatrix} = \underline{U}(z)$

If the $\underline{u}_1^{\text{[redacted]}} \; \underline{u}_n^{\text{[redacted]}}$ are linearly dependent then $W(\underline{u}_1^{\text{[redacted]}}(z), \dots, \underline{u}_n^{\text{[redacted]}}(z), z) = \det[\underline{u}_1^{\text{[redacted]}}(z), \dots, \underline{u}_n^{\text{[redacted]}}(z)]$ would be zero for all z. Thus W being nonzero somewhere is the indication that these are linearly independent solutions.

Since $\frac{d}{dz}\underline{u}_i(z) = -\underline{P}(z)\,\underline{u}_i(z)$,

Consider $\underline{u}_i(z+h)$ for small h.

$$\underline{u}_i(z+h) = \underline{u}_i(z) - h\underline{P}(z)\,\underline{u}_i(z) + O(h^2)\,\underline{u}_i(z)$$

So
$$[\underline{u}_1(z+h) \cdots \underline{u}_n(z+h)] = \left(\underline{I} - h\underline{P}(z) + O(h^2)\right)[\underline{u}_1(z) \cdots \underline{u}_n(z)]$$
$$= \left[\underline{I} - h\underline{P}(z) + O(h^2)\right]\underline{U}(z)$$

Taking det

$$W\big|_{z+h} = \det\left[\underline{I} - h\underline{P}(z) + O(h^2)\right] W\big|_z$$
$$= \left(1 - h\,\mathrm{tr}\,\underline{P}(z) + O h^2\right) W\big|_z$$
$$\frac{dW}{dz} = -\mathrm{tr}(\underline{P}(z))\,W(z)$$

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Note that

$$\det \begin{bmatrix} 1-hP_{11}(z) & -hP_{12}(z) & \cdots & -hP_{1n}(z) \\ -hP_{21}(z) & 1-hP_{22}(z) & \cdots & -hP_{2n}(z) \\ \vdots & & & \vdots \\ -hP_{n1}(z) & -hP_{n2}(z) & \cdots & -hP_{nn}(z) \end{bmatrix}$$

$$= (1-hP_{11}(z))\cdots(1-hP_{nn}(z)) + O(h^2)$$

$$= 1 - h\,(P_{11}(z) + \cdots + P_{nn}(z)) + O(h^2)$$

$$= 1 - h\,\mathrm{tr}(\underline{P}(z)) + O(h^2)$$

Since [redacted] W is a scalar satisfying a first order homogeneous equation, we can solve it exactly

$$W(z) = e^{-\int_{z_0}^{z} \mathrm{tr}\,\underline{P}(\xi)\,d\xi}\, W(z_0)$$

If we restricted ourselves to $P(z) = -A$, then

$-\mathrm{tr}\,\underline{P}(z) = \mathrm{tr}\,A = \lambda_1 + \cdots + \lambda_n$.

In the diagonalizable case

with real argument, "volumes" change like $e^{\sum \lambda_i t}$.

The change of Wronskian just captures that.

$\underline{U}(z;z_0) = [\,\underline{u}_1(z;z_0) \cdots \underline{u}_n(z;z_0)\,]$, [redacted] is called a fundamental solution if $L[\underline{u}(\cdot)] = 0$ and $\underline{U}(z_0;z_0) = \underline{\underline{I}}$

Now we take on nth-order differential equation!

Linear nth order differential equation is defined by the operator L

$$L[u](z) = u^{(n)}(z) + p_1(z)\,u^{(n-1)}(z) + p_2(z)\,u^{(n-2)}(z) \cdots + p_{n-1}(z)\,u'(z) + p_n(z)\,u(z)$$

$L[u] = 0$ is the homogeneous equation

$L[u](z) = f(z)$ is the inhomogeneous equation.

If v [redacted] is a special solution, so that

$$L[v](z) = f(z)$$

and u [redacted] is a solution of the homogeneous equation

$$L[u+v](z) = f(z)$$

In fact, any solution of the inhomogeneous equation can be written this way since u_{inhom} [redacted] $- v$ [redacted] is a solution to the homogeneous equation.

Now, let us write this equation as a system of first order equations.

With $\underline{u} = \begin{pmatrix} u \\ u' \\ u'' \\ \vdots \\ u^{(n-2)} \\ u^{(n-1)} \end{pmatrix}$

$$\frac{d}{dz}\begin{pmatrix} u \\ u' \\ \vdots \\ u^{(n-2)} \\ u^{(n-1)} \end{pmatrix} + \begin{pmatrix} 0 & 1 & & & \\ 0 & 0 & \ddots & & \\ & & \ddots & \ddots & \\ & & 0 & 0 & 1 \\ p_n & p_{n-1} & \cdots\ p_3 & p_2 & p_1 \end{pmatrix} \begin{pmatrix} u \\ u' \\ \vdots \\ u^{(n-2)} \\ u^{(n-1)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f \end{pmatrix}$$

For simplicity, we will [redacted] take $f = 0$, but it is not essential.

We define $W(u_1, \ldots, u_n) = \begin{vmatrix} u_1 & u_2 & \cdots & u_n \\ u_1' & u_2' & & u_n' \\ \vdots & & & \vdots \\ u_1^{(n-1)} & u_2^{(n-1)} & \cdots & u_n^{(n-1)} \end{vmatrix}$

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Note that $\underline{P}(z) = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & \ddots & 1 \\ p_n(z) & - & p_2(z) & p_1(z) \end{pmatrix}$ So $\mathrm{tr}\,\underline{P}(z) = p_1(z)$

So $\frac{d}{dz} W + p_1(z) W = 0$

$$W(z) = e^{-\int_{z_0}^{z} p_1(\zeta)\, d\zeta}\, W(z_0)$$

The fundamental soln $u_k(\cdot\,; z_0)$ with $u_k^{(n-1)}(z_0; z_0) = \delta^{kn}$.

Power Series Solution:

$u''(z) + z\, u(z) = 0$ Airy equation

Try $u(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ Call $a_0 = 1$.

[redacted] $(2\cdot 1 a_2 + 3\cdot 2 a_3 z + 4\cdot 3 a_4 z^2 + \cdots) + z(a_0 + a_1 z + a_2 z^2 + \cdots) = 0$

$3\cdot 2 a_3 + a_0 = 0$, $4\cdot 3 a_4 + a_1 = 0$, $5\cdot 4 a_5 + a_2 = 0$, $6\cdot 5 a_6 + a_3 = 0$,

$$a_{n+3} = -\frac{a_n}{(n+3)(n+2)}$$

$\left|\frac{a_{3m+3}}{a_{3m}}\right| = \frac{1}{3(m+1)(3m+2)}$ absolutely conv.

$u(0) = 1$ $u'(0) = a$ Note that $u''(0) = 0$

$$u(z) = 1 - \frac{1}{3\cdot 2} z^3 + \frac{1}{6\cdot 5\cdot 3\cdot 2} z^6 - \cdots$$
$$+ az - \frac{a}{4\cdot 3} z^4 + \frac{a}{7\cdot 6\cdot 4\cdot 3} z^7 - \cdots$$

$\left\} \begin{array}{l} 1 + \sum_{n=1}^{\infty} (-1)^n \frac{(3n-2)(3n-5)\cdots}{(3n)!} z^{3n} \\ + az\left[1 + \sum_{n=1}^{\infty} (-1)^n \frac{(3n-1)(3n-4)\cdots 2}{(3n+1)!} z^{3n}\right] \end{array}\right.$

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$\left|\frac{a_{3m+4}}{a_{3m+1}}\right| = \frac{1}{3(m+1)(3m+4)}$ abs. conv

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Note that [redacted] the radius of convergence is ∞.

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The coefficients are analytic everywhere and so is u.