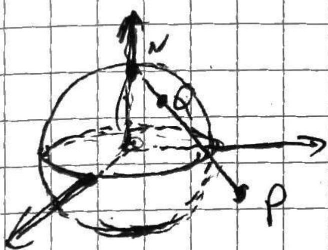
Point at Infinity

We can add the point ∞ to $\mathbb{C}$: $\Sigma = \mathbb{C} \cup \{\infty\}$ is the extended complex plane. Σ can be regarded as a sphere

Consider $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$



$z = x + iy$

$x_1 = \frac{2x}{x^2+y^2+1}$ $x_2 = \frac{2y}{x^2+y^2+1}$, $x_3 = \frac{x^2+y^2-1}{x^2+y^2+1}$

N is the point $(0, 0, 1)$. The whole complex plane maps to $S^2 - \{N\}$. We could make the point ∞ correspond to the point N, setting up a correspondence between S^2 and Σ. Use the topology on S^2 induced from $\mathbb{R}^3$ to setup a topology on Σ.

Note that the [redacted] neighborhood of $+\infty$ could be brought to the neighborhood of 0 by the conformal map $w = \frac{1}{z}$

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For [redacted] example $f(z) = z^n$ goes to $g(w) = \frac{1}{w^n}$, essentially a pole near origin.

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Now let us take on rational functions and there behavior at ∞.

Def f is rational function if $f = \frac{p}{q}$ where p and q are polynomials.

If $p(z)$ has ~~of~~ degree m and $q(z)$, degree n

(i) if $n > m$, f has a zero of order $n-m$ at infinity

(ii) " $n = m$, f is analytic at $+\infty$

(iii) " $n < m$, f has pole of order $m-n$ at infinity

Essentially $\dfrac{z^m(a_m + a_{m-1}/z + \cdots\ )}{z^n(b_n + b_{n-1}/z + \cdots\ )}$

goes to $\omega^{n-m}\left[\frac{a_m}{b_n} + \frac{a_m}{b_n}\left(\frac{a_{m-1}}{a_m} - \frac{b_{n-1}}{b_n}\right)\omega + O(\omega^2)\right]$

with $z = \frac{1}{\omega}$

Def. A function f is meromorphic in a region R if the function is analytic in $R - A$ where A is a finite set and each point in A is pole of f.

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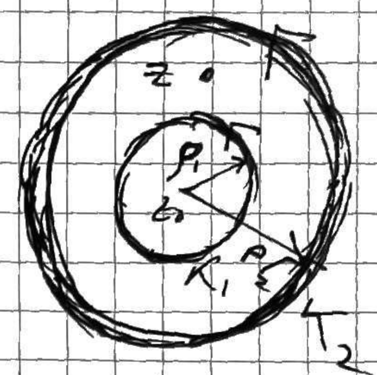
Laurent Series

Consider f which is analytic in an annulus
$R = \{ z \mid \rho_1 < |z - z_0| < \rho_2 \}$



$$f(z) = \frac{1}{2\pi i}\left\{ \oint_{K_2} \frac{f(\xi)}{\xi - z}\, d\xi - \oint_{K_2} \frac{f(\xi)}{\xi - z}\, d\xi \right\}$$

$$\frac{1}{2\pi i}\oint_{K_2} \frac{f(\xi)}{\xi - z}\, d\xi = \frac{1}{2\pi i}\oint_{K_2} \frac{f(\xi)}{(\xi - z_0)\left(1 - \frac{z - z_0}{\xi - z_0}\right)}\, d\xi$$

$$= \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where $a_n = \frac{1}{2\pi i} \oint_{K_2} \frac{f(\xi)}{(\xi - z_0)^{n+1}}\, d\xi$

$$\frac{1}{2\pi i}\oint_{K_1} \frac{f(\xi)}{\xi - z}\, d\xi = -\frac{1}{2\pi i}\oint_{K_1} \frac{f(\xi)}{z - \xi}\, d\xi = -\frac{1}{2\pi i}\oint_{K_1} \frac{f(\xi)}{(z - z_0)\left(1 - \frac{\xi - z_0}{z - z_0}\right)}\, d\xi$$

$$= -\sum_{n=0}^{\infty} b_n (z - z_0)^{-n-1}$$

With $b_n = \frac{1}{2\pi i} \oint_{K_1} f(\xi)(\xi - z_0)^n\, d\xi$

So $f(z) = \sum_{n=0}^{\infty} \left[a_n (z - z_0)^n + b_n (z - z_0)^{-n-1} \right]$

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Read and Understood By

This series is absolutely convergent. If $|f(\xi)| \le M_2$ on K_2 and $|f(\xi)| \le M_1$ on K_1, then $|a_n| \le \frac{M_2}{\rho_2^{n+1}}$, and $|b_n| \le M \rho_1^{n+1}$

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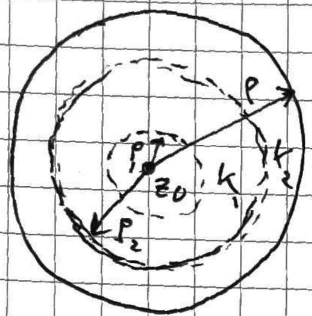
Now consider f defined on a disc of radius ρ centered centered around z_0, $D(z_0, \rho)$, except at the point z_0. So the region R is $D(z_0, \rho) - \{z_0\}$ where the function f is analytic, with possibly an isolated singularity at z_0.



We could set up $0 < \rho_1 < \rho_2 < \rho$ and in the region where

$$\rho_1 < |z - z_0| < \rho_2,$$

we can use the Laurent series.

Note that a_n and b_n expressions do not change, in this case, if ρ_1 and ρ_2 are changed with the inequalities $0 < \rho_1 < \rho_2 < \rho$ satisfied.

So we can bring ρ_2 as close to ρ as possible and ρ_1 as close to zero as possible and show that the expression of Laurent series is valid in all of $D(z_0, \rho) - \{z_0\}$.

We will often call $b_n = a_{-n-1}$

Then $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$

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$$f(z) = \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

Def: The coefficient a_{-1} is called the residue of f at z_0.

We will use the notation $\mathrm{Res}_{z=z_0} f(z) = a_{-1}$

Note that $\oint_C f(z)\,dz$ with a contour around z_0 is given by $2\pi i\, a_{-1}$, the so-called Cauchy residue theorem.

Def: The function R, with $R(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$, is called the regular part of f.

Def: The function S, with $S(z) = \sum_{n=1}^{\infty} \frac{a_{-n}}{(z-z_0)^n}$, is called the singular part of f.

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Comments on Weierstrass Factorization Theorem and Mittag-Leffler Theorem

Weierstrass factorization theorem is concerned with entire function with zeros written in terms of an entire function without zeros and the zeros and their multiplicity's

<u>Weirstrass factorization theorem</u>: If F is an entire function with zeros $\{z_k\}$ with $z_k \neq 0$ for all $k \in N$, $F(0) \neq 0$, then there is an entire function H, with no zeros in the complex plane, so that

$$F(z) = H(z) \prod_{k=1}^{\infty} \left[\left(1 - \frac{z}{z_k}\right) e^{\gamma_k(z)} \right]$$

with γ_k's being polynomials of the form

$$\gamma_k(z) = 1 \qquad (m_k = 0)$$

$$= \left(\frac{z}{z_k}\right) + \frac{1}{2}\left(\frac{z}{z_k}\right)^2 + \cdots + \frac{1}{m_k}\left(\frac{z}{z_k}\right)^{m_k} \quad \text{for } m_k \neq 0$$

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with $\{m_k\}$ being a sequence of nonnegative integers.

[redacted]

Example: $$\frac{\sin z}{z} = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2\pi^2}\right)$$

The Mittag-Leffler theorem provides a representation of meromorphic functions in [redacted] a region $\mathcal{R}$ with A being the set of poles $\{z_1, z_2, \dots, z_m\}$. Around each pole z_k we could look at a Laurent series and look at the singular part $S_k(z) = \sum_{n=1}^{p_k} \frac{a_{k,n}}{(z-z_k)^n}$. Note that p_k is the order of the pole.

Mittag-Leffler theorem: For $z \in \mathcal{R} - A$

$$f(z) = \sum_{k=1}^{m} S_k(z) + Q(z)$$

where Q is a function analytic in $\mathcal{R}$.

For infinitely many poles $\{z_k\}$ with A not having any limit point in $\mathbb{C}$,

$$f(z) = \sum_{k=1}^{\infty}\left(S_k(z) - \sigma_k(z)\right) + E(z)$$

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where $\sigma_k(z)$ are polynomials so that the series converges and E is an entire function.

[These two theorems are related: consider $f(z) = \frac{F'(z)}{F(z)}$]

Calculus of Residues

[illegible]

Cauchy residue theorem: Let R be a closed simply connected region bounded by a closed contour C. Let f be function analytic in $R - \{z_1, \cdots, z_N\}$, with z_k being interior points of R for $k = 1, \cdots, N$. With anticlockwise orientation being chosen on C,

$$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{N} \operatorname{Res}_{z=z_k} f(z)$$

Informal Proof:

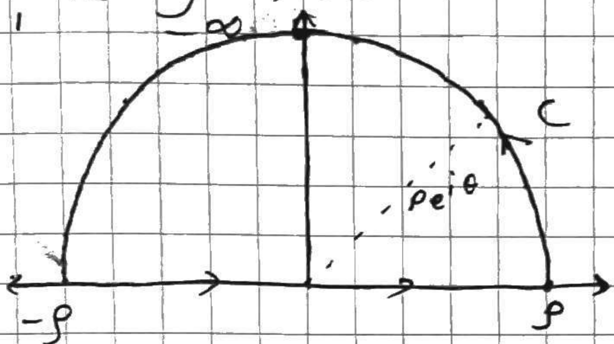
$$\oint_{C - C_1 - \cdots - C_N} f(z)\,dz = 0$$

$$\oint_C f(z)\,dz = \sum_{k=1}^{N} \oint_{C_k} f(z)\,dz$$

C_k is a small loop around z_k $\oint_{C_k} f(z)\,dz = 2\pi i \operatorname{Res}_{z=z_k} f(z)$

Using Cauchy residue theorem for doing real integral

$$I_1 \equiv \int_{-\infty}^{\infty} f(x)\,dx$$



If $z f(z) \to 0$ as $z \to \infty$ in the upper half plane then we can add an arc $\{z \mid z = \rho e^{i\theta},\ \theta \in [0,\pi]\}$ to the interval [redacted] $\{x + i0 \mid x \in [-\rho, \rho]\}$. As $\rho \to \infty$ The integral on the real line [redacted] goes to I_1, and the integral on the arc goes to zero.

$$\oint_C f(z)\,dz = 2\pi i \sum_{k} \operatorname{Res}_{z=z_k} f(z)$$

Where z_k are the isolated singularities in the upper half plane.

Example(1) $I(\alpha) = \int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{x^2+a^2}\,dx$, [redacted] $a > 0$
(Add an arc)

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When $\alpha \geq 0$

On the upper half plane $z = x + iy$, $y \geq 0$

$$\left| z \frac{e^{i\alpha z}}{z^2 + a^2} \right| = \frac{|z| \, e^{-\alpha y}}{|\cancel{z}^2 + a^2|} \leq \frac{\text{[redacted]} \, e^{-\alpha y}}{|z|(1 - \frac{a^2}{|z|^2})}$$

for $|z| > a$

As $|z| \to \infty$ $\left| z \frac{e^{i\alpha z}}{z^2 + a^2} \right| \to 0$.

$$f(z) = \frac{e^{i\alpha z}}{z^2 + a^2} = \frac{e^{i\alpha z}}{(z + ia)(z - ia)}$$ has two singularities

$z = \pm ia$ with $\operatorname{Res}_{z = \pm ia} f(z) = \frac{e^{\mp \alpha a}}{\text{[redacted]}(\pm 2ia)}$

The one on upper half plane is $\frac{e^{-a\alpha}}{2ia}$

So $2\pi i \operatorname{Res}_{z = ia} f(z) = \frac{\pi}{a} e^{-a\alpha}$

When $\alpha < 0$, We need to put an arc in the lower half plane

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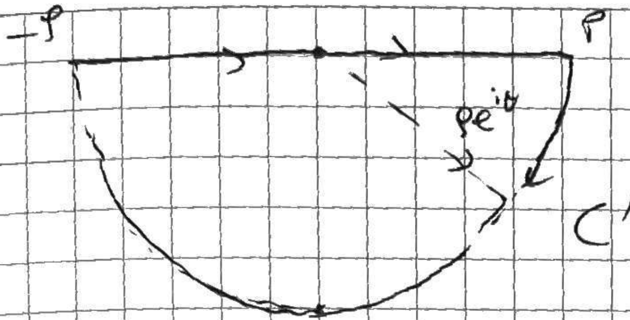

The arc $\{z = \rho e^{i\theta} \mid \theta \in [-\pi, 0]\}$

Once more, the contribution of the arc vanishes when $\rho \to \infty$.

Notice that this contour is a clockwise on.

So $\displaystyle\oint_{C'} f(z)\,dz = -2\pi i \sum_{\ell} \operatorname{Res}_{z=z_\ell} f(z)$

Where z_ℓ is in the lower half plane

$$\operatorname{Res}_{z=-ia} f(z) = -\frac{e^{a\alpha}}{2ia}$$

So $\displaystyle\oint_{C'} f(z)\,dz = \frac{\pi}{a} e^{a\alpha}$

So far $\alpha \in \mathbb{R}$ $\displaystyle I(\alpha) = \int_{-\infty}^{\infty} \frac{[redacted]\, e^{i\alpha x}\,dx}{x^2 + a^2} = \frac{\pi e^{-a|\alpha|}}{a}$

Example 2) $\displaystyle F(a,b) = \int_0^{2\pi} \frac{d\theta}{a - b\cos\theta}$ with $|b| < |a|$
(Find a transformation)

Try $z = e^{i\theta}$

$$\oint_C \frac{1}{a - \frac{1}{2}b\left(z + \frac{1}{z}\right)} \frac{dz}{iz} = 2i \int_C \frac{dz}{bz^2 - 2az + b}$$

Where are the singularities. $b\left(z - \frac{a}{b}\right)^2 = \frac{a^2}{b} - b \Rightarrow z_{1,2} = \frac{a}{b} \pm \sqrt{\left(\frac{a}{b}\right)^2 - 1}$

Note that $z_1 z_2 = 1$ $|z_1| > 1$, $|z_2| < 1$

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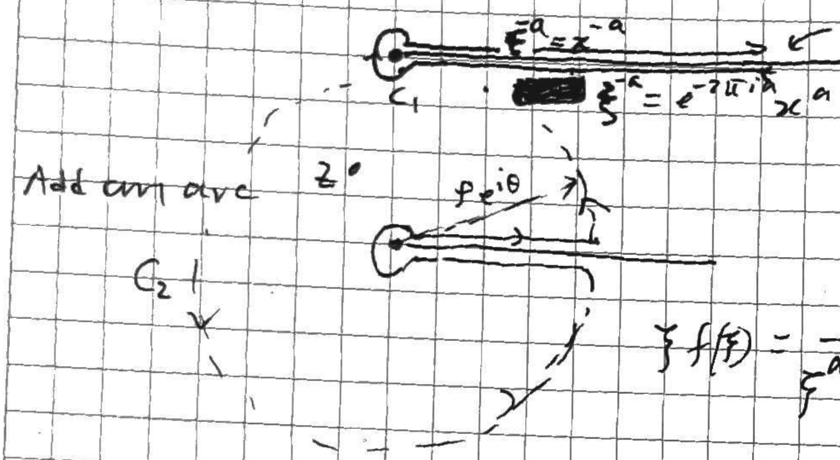
$2\pi i \operatorname{Res}_{z=z_2} \frac{1}{(z-z_1)(z-z_2)}$

$$F(a,b) = 2i \oint \frac{dz}{b(z-z_1)(z-z_2)} = \frac{2i}{b} \frac{2\pi i}{z_2 - z_1} = \frac{-4\pi}{b\left(-2\sqrt{\frac{a^2}{b^2}-1}\right)}$$

$$= \frac{2\pi}{\sqrt{a^2-b^2}}$$

Example 3)
(Deal with a branch cut)

$$I(a) = \int_0^\infty \frac{dx}{x^a (x-z)} \qquad 0<|a|<1$$



$$\frac{1}{1-e^{-2\pi i a}} \int_{C_1} \frac{d\xi}{\xi^a (\xi - z)}$$

$$\xi f(\xi) = \frac{\xi}{\xi^a (\xi - z)} \longrightarrow 0 \quad \text{as } |\xi| \to \infty$$

Let $C = C_1 + C_2$

$$F(a,b) = \lim_{\rho\to\infty} \oint_C \frac{1}{1-e^{-2\pi i a}} \frac{d\xi}{\xi^a (\xi - z)}$$

Where is the singularity? At $\xi = z$

$$F(a,b) = \frac{1}{1-e^{-2\pi i a}} \frac{2\pi i}{z^a} = \frac{\pi}{\sin \pi a} (-z)^{-a}$$

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Example 4) $I = \int_0^{\infty} \frac{dx}{1+x^3}$

(Invent an integrand)

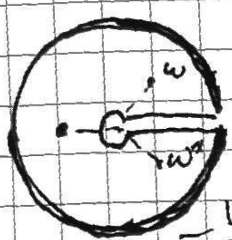
For $\int_0^{\infty} \frac{dx}{1+x^{2m}}$ could be [redacted] rewritten as $\frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{1+x^{2m}} = \frac{1}{2}\oint_C \frac{dz}{1+z^{2m}}$

Cannot do this here!

Use the branch cut trick with $\ln z$ (think of it as a→0 limit)

$\ln z = \ln x$
[redacted] $\ln z = \ln x + 2\pi i$

$$\int_0^{\infty} \frac{dx}{1+x^3} = -\frac{1}{2\pi i}\int_C \frac{\ln z}{1+z^3}\,dz$$



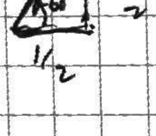
Poles at $z = (-1)^{\frac{1}{3}} \Rightarrow -1,\ \omega = e^{\pi i/3},\ \omega^* = e^{-\pi i/3} = e^{5\pi i/3}$

$\ln(-1) = \pi i \qquad \ln\omega = \frac{\pi i}{3} \qquad \ln\omega^* = \frac{5\pi i}{3}$

$$-\frac{1}{2\pi i}\oint_C \frac{\ln z}{(z+1)(z-\omega)(z-\omega^*)}\,dz = -\frac{2\pi i}{2\pi i}\left[\frac{\ln(-1)}{(-1-\omega)(-1-\omega^*)} + \frac{\ln\omega}{(\omega+1)(\omega-\omega^*)} + \frac{\ln\omega^*}{(\omega^*+1)(\omega^*-\omega)}\right]$$

$$= \frac{i\pi}{3}\left\{\frac{5}{(\omega-\omega^*)(\omega^*+1)} - \frac{3}{(\omega+1)(\omega^*+1)} - \frac{1}{(\omega-\omega^*)(\omega+1)}\right.$$

BTW $\omega = e^{\frac{i\pi}{3}} = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2}(1+i\sqrt{3})$



$\omega^* = \frac{1}{2}(1-i\sqrt{3})$

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Comments on some Special Functions

Gamma function:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt \qquad \text{for } \operatorname{Re} z > 0$$

On other parts of complex plane Γ can be define via analytic continuation

$$\Gamma(n+1) = \int_0^\infty t^n e^{-t}\, dt = n! \qquad \text{when } n = 0, 1, 2, \ldots$$

Near $z=0$ The integral diverges near $t=0$

$$\int_0^M t^{z-1}\, dt \sim \frac{t^z}{z}\Big|_0^M \sim \frac{M^z}{z} \sim \frac{1 + z\ln M + \cdots}{z} \sim \frac{1}{z}$$

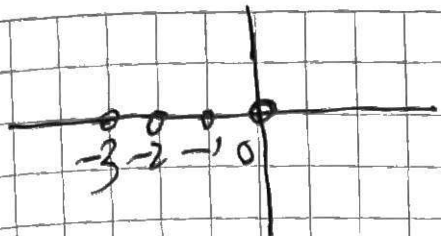
This suggests a simple pole.

Also $$\Gamma(z+1) = \int_0^\infty t^z e^{-t}\, dt = -\int_0^\infty \frac{d}{dt}\left(t^z e^{-t}\right) dt + z\int_0^\infty t^{z-1} e^{-t}\, dt$$

$$= 0 + z\Gamma(z) \qquad \text{for } \operatorname{Re} z > 0$$

So $\boxed{z\Gamma(z) = \Gamma(z+1)}$

If we use this to continue $\Gamma(z)$ to $\operatorname{Re} z \leq 0$ Continued on Page



$\Gamma(z)$ is analytic everywhere

$\Gamma(z) = \frac{1}{z}\Gamma(1+z)$ — $\Gamma(1) = 1$ and $\Gamma(1+z)$ is [redacted] well behaved for small z

$\Gamma(z) \sim \frac{1}{z}$ for [redacted] small z

Other poles are at $-n$, with the residue being $(-1)^n/n!$

Check: $\Gamma(-1+\epsilon) = \frac{1}{-1+\epsilon}\Gamma(\epsilon) \sim \frac{1}{-1}\frac{1}{\epsilon}$

$\Gamma(-2+\epsilon) = \frac{1}{-2+\epsilon}\Gamma(-1+\epsilon) \sim \frac{1}{-2}\frac{1}{-1}\frac{1}{\epsilon}$ etc etc

Note that $\Gamma(z)\,\Gamma(1-z)$ has poles on every integer. At [redacted] $n>0$ $\Gamma(n+\epsilon) = (n-1)! + O(\epsilon)$

$\Gamma(1-n-\epsilon) = \frac{(-1)^{(n-1)}}{(n-1)!}\frac{1}{-\epsilon} + O(1)$

For $n \leq 0$ $\Gamma(n+\epsilon) = \frac{(-1)^n}{(-n)!}\frac{1}{\epsilon} + O(1)$

$\Gamma(1-n-\epsilon) = (-n)!$ [redacted] $+ O(\epsilon)$

So the pole at n has residue $(-1)^n$. $f(z) = \frac{\pi}{\sin \pi z}$ has the same pole structure

Turns out that $\boxed{\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin \pi z}}$

Beta Function: $B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt$ with the integral defined for $\mathrm{Re}\,\alpha > 1$, $\mathrm{Re}\,\beta > 1$

In that range $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

We can use that to provide analytical continuation of $B(\alpha,\beta)$ to $\mathbb{C}^2$ except at [redacted] points where α, β have negative integer values.