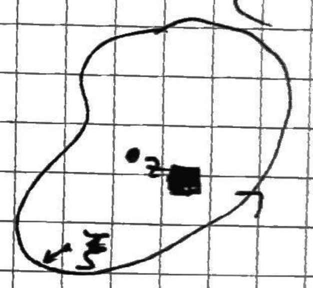
PROJECT

Cauchy's Integral Formula

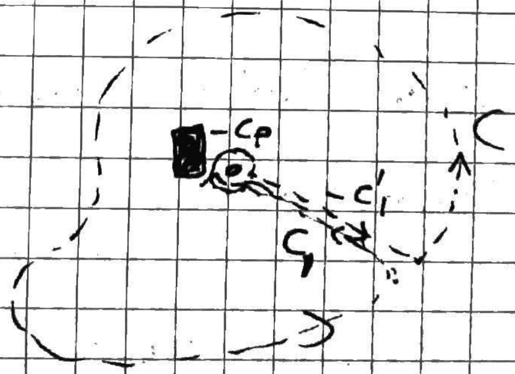
If f is (an analytic function) defined on a closed simply connected region R, whose boundary is a closed contour C, then Cauchy integral formula gives us the value of f at z, which is an interior point of R, in terms of [redacted] a closed contour integral on the boundary.



$$\boxed{\oint_C \frac{f(\xi)}{\xi - z} = 2\pi i\, f(z)}$$

C has to run anticlockwise.

Informal proof:



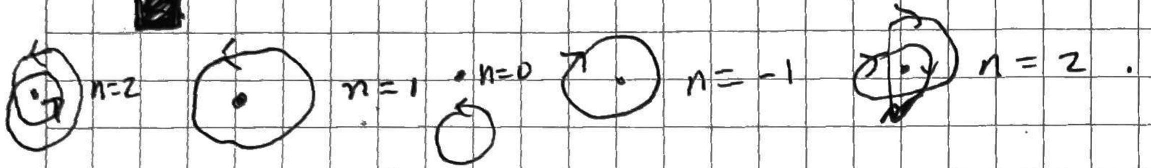
$\{z+\rho e^{i\theta}$
$\theta\in[0,2\pi]$

$$\int_{C+C_1-C_1'-C_\rho} f(\xi)/(\xi-z) = 0 \Rightarrow \int_C \frac{f(\xi)}{\xi - z}\,dz = \int_{C_\rho} \frac{f(\xi)}{\xi - z}\,dz$$

$$= \int \frac{f(z+\rho e^{i\theta})}{\rho e^{i\theta}}\,\rho i e^{i\theta}\,d\theta = i\int f(z+\rho e^{i\theta})\,d\theta \to 2\pi i f(z) \text{ as } \rho\to 0$$

Essentially, values of analytic functions in R_0 are determined by values on ∂R_0.

Note: For a closed contour C and a point z, we can define an integer $n(C,z)$ given by $\frac{1}{2\pi i}\int_C \frac{d\xi}{\xi - z}$ when C does not pass through z.

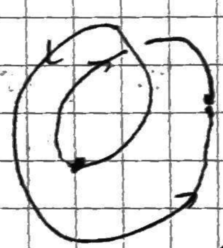


This number is called the winding number of the closed contour C. The general Cauchy integral formula for a closed contour in a simply connected region is that

$$\int_C \frac{f(\xi)\,d\xi}{\xi - z} = 2\pi\, n(C,z)\, i\, f(z)$$

To see why $\int_C \frac{d\xi}{\xi - z}$ is a multiple of $2\pi i$

Consider defining $h(t) = \int_0^t \frac{\xi'(t')}{\xi(t') - z}\,dt' = \ln\left(\frac{\xi(t) - z}{\xi(0) - z}\right)$



$\xi : [0, \alpha] \to \mathbb{C}$

So $e^{h(t)} = (\xi(t) - z)/(\xi(0) - z)$

For a closed contour $\xi(\alpha) = \xi(0)$. So $e^{h(\alpha)} = e^{h(0)}$

So $h(\alpha) - h(0) = 2\pi n i$ with $n \in \mathbb{Z}$

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We have that $f(z) = \frac{1}{2\pi i}\int_C \frac{f(\xi)}{\xi - z}$ for a closed contour of winding number 1 (around z)

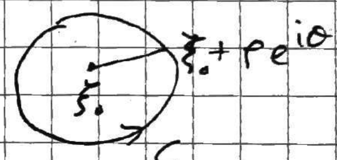
Differentiating in z multiple times under both sides (differentiation under integration can be done [redacted] as $\frac{d}{dz}\frac{f(\xi)}{(\xi-z)^m}$ is continuous on the contour):

$$\boxed{f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\xi)}{(\xi - z)^{n+1}}}$$

Note that this implies all derivatives of an analytic function exist and are analytic.

Let us see a few examples.

a) $f(\xi) = (\xi - \xi_0)^m$



$\theta \in [0, 2\pi]$

[redacted]

$$\oint_C \frac{f(\xi)}{(\xi - z)^{n+1}} = i\int_0^{2\pi} \frac{(\rho e^{i\theta})^m}{(\rho e^{i\theta})^{n+1}} \rho e^{i\theta} d\theta = \rho^{m-n} i \int_0^{2\pi} e^{i(m-n)\theta} d\theta$$

$$= 2\pi \rho^{m-n} \delta_{m,n} i = 2\pi i \delta_{m,n}$$

Note that $f^{(n)}(\xi) = m(m-1)\cdots(m-n+1)(\xi - z)^{m-n}$ $m \geq n$
$= 0$ $m < n$

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$f^{(n)}(z) \neq 0$ only when $m = n$. $f^{(m)}(z) = m!$

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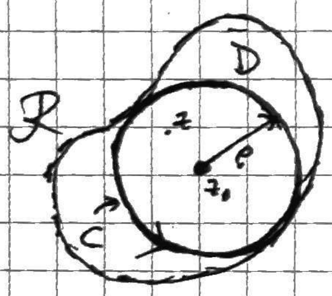
We can use the representation of derivatives to [redacted] derive something about [redacted] Taylor expansion of analytic functions.

Remember that if $\{u_n\}$ are a sequence of functions and $\sum u_n$ converged uniformly to its limit in some domain, then one can do integration term by term and then add. namely $\int \sum u_n = \sum \int u_n$

Thm: If f in analytic in a region R and the closed disc $\bar{D} = \{z \,|\, |z-z_0| \leq \rho\}$ [redacted] in a subset of R, then the Taylor series for f [redacted] around z_0 converges in the open [redacted] disc

$D = \{z \,|\, |z-z_0| < \rho\}$.

Proof: [redacted] $f(z) = \oint_C \frac{f(\xi)\,d\xi}{\xi - z}$



Where C is parametrized as $\{z_0+\rho e^{i\theta} \,|\, \theta\in[0,2\pi]\}$

Let us expand $\frac{1}{\xi - z} =$ [redacted] $\frac{1}{\xi - z_0 + z_0 - z} =$ [redacted] $\frac{1}{(\xi - z_0)\left[1-\frac{z-z_0}{\xi-z_0}\right]}$

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$$= \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(\xi - z_0)^{n+1}}$$

This expansion converges because $\left| \frac{z-z_0}{\xi - z_0} \right| < 1$

Also, for any such z, [redacted] the [redacted] convergence is uniform, in that, it does not depend on ξ, since $|\xi - z_0| = \rho$

and $\left| \sum_{n=N}^{\infty} \frac{(z-z_0)^n}{(\xi-z_0)^{n+1}} \right| \leq \frac{|z-z_0|^N}{\rho^{N+1}} \frac{1}{1 - \frac{|z-z_0|}{\rho}}$.

So $$f(z) = \sum_{n=0}^{\infty} (z-z_0)^n \oint_C \frac{f(\xi)\, d\xi}{(\xi - z_0)^{n+1}}$$ (term by term integration)

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z-z_0)^n$$

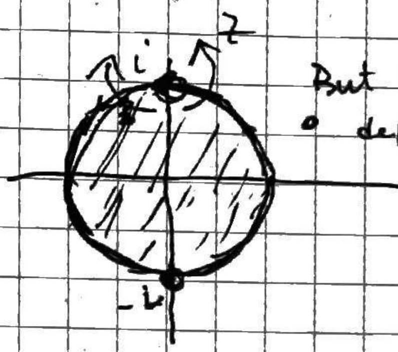
$\boxed{\text{QED}}$

Examples: 1) $f(z) = \frac{1}{1+z^2}$

Expand around $z = 0$

$f(z) = 1 - z^2 + z^3 - z^4 + \cdots$

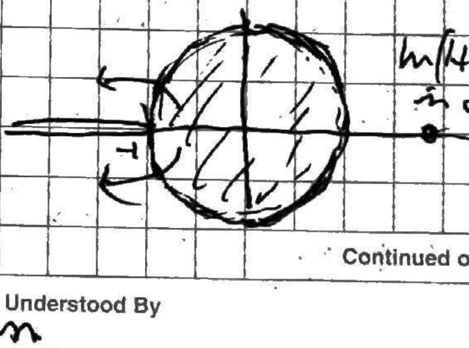
Radius of convergence 1.



But $\frac{1}{1+z^2}$ defined here!

2) $f(z) = \ln(1+z)$

$f(z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$



$\ln(1+z)$ is defined here!

Radius of convergence 1

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The fact that f can be Taylor expanded at any interior point of $\mathcal{R}$ has some very powerful consequences.

BTW, differentiability implies continuity for complex functions as well, but you already knew that!

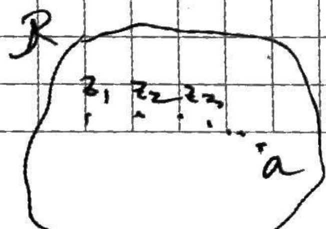
Also, remember the definition of a limit point of a set.

Thm: Let f be analytic on a region $\mathcal{R}$ and let $Z(f) = \{ z \in \mathcal{R} \mid f(z) = 0 \}$, namely the set of zeros. Then, either $Z(f) = \mathcal{R}$ or $Z(f)$ has no limit point in $\mathcal{R}$.

[redacted]

Proof: Let A be the limit points of $Z(f)$ inside $\mathcal{R}$. Since f is continuous $A \subset Z(f)$.

The reason is that we could set up a sequence $\{z_n\}$ inside $Z(f)$ that converges to $a \in A$ with the sequence and its limit all in $\mathcal{R}$.



$f(z_n) = 0$ [redacted] for all $n \in \mathbb{N}$. Since f is continuous and $z_n \to a$ $f(z_n) \to f(a)$ and $f(a) = 0$

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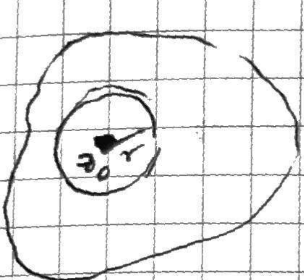
Now consider $z_0 \in Z(f)$. Since f is analytic on R, there is a radius r so that one can Taylor expand $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ for $z \in D(z_0, r)$ [redacted] $\subset R$. $D(z_0, r) \equiv \{z' \in \mathbb{C} \mid |z' - z_0| < r\}$



Since $f(z_0) = 0$, $a_0 = 0$,

either all $a_n = 0 \Rightarrow f(z) = 0$ inside $D(z_0, r)$

or there is a smallest m for which $a_m \neq 0$

In the first case $D(z_0, r) \subset A$ and z_0 is an interior point of A. In the other case, we can show that z_0 is an isolated point and therefore does not belong to A.

We [redacted] do that by defining $g(z) = \frac{f(z)}{(z - z_0)^m}$ at $z \neq z_0$

$= a_m$ at $z = z_0$

In $D(z_0, r)$, $g(z) = \sum_{n=0}^{\infty} a_{n+m} (z - z_0)^n$. g is analytic everywhere in R and $g(z_0) \neq 0$

By continuity of analytic functions $g(z) \neq 0$ in some neighborhood for $z_0 \Rightarrow f(z) \neq 0$ except at $z = z_0$ in that neighborhood

So z_0 is an isolated point in R and $z_0 \notin A$.

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So all points in A are interior points $\Rightarrow A$ is open!

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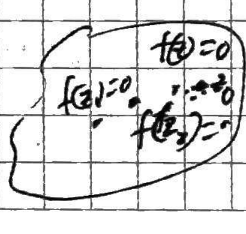
A is a set of limit points in R. So $R-A$ is open. [redacted] If there was a point b in $R-A$ which had a point of A in any neighborhood, we can set up a sequence of points in $Z(f)$ converging to b.

[redacted]

Since R is connected and A and $R-A$ are disjoint and open, one of them has to be null. So, either $A=R$ or $A=\phi$. Since $A \subseteq Z(f) \subseteq R$. Either $Z(f)=R$ or [redacted] $A=\phi$

$\boxed{\text{QED}}$

Corollary 1 [redacted]: If there is a sequence $\{z_k\}$ in a region R with a limit point $z_0 \in R$, and an analytic function on R, f, is zero at each point z_k, it is zero in all of R.
(Thm 4.1 in the book)



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Corollary 2: Let [redacted] R_1 and R_2 are overlapping regions, namely $R_1 \cap R_2 \neq \phi$, with f_1 analytic in R_1 and f_2 analytic on R_2. Assume that $R_1 \cap R_2$ is connected. Let $f_1(z) = f_2(z)$ in any neighborhood of $z_0 \in R_1 \cap R_2$

Then there is an [redacted] analytic function f [redacted] defined over $R_1 \cup R_2$, [redacted] so that the restrictions of f to R_i is f_i, $i = 1, 2$.

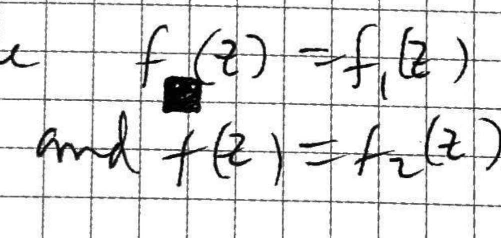
Proof: $R_1 \cap R_2$ is open, nonempty and connected. The previous theorem says $f_1(z) - f_2(z) = 0$ everywhere in $R_1 \cap R_2$, if it is zero in an open disc inside $R_1 \cap R_2$.

Define f[redacted]$(z) = f_1(z)$ on R_1

and $f(z) = f_2(z)$ on R_2.

$\boxed{QED}$

One can use Taylor expansion is overlapping discs to continue a function.



$\Rightarrow$ Analytic Continuation of (f_0, D_0)

$(f_0, D_0), (f_1, D_1), (f_2, D_2), \dots, (f_n, D_n)$

(f_i, D_i) function element.

with $D_i \cap D_{i+1} \neq \phi$ and $f_i(z) = f_{i+1}(z)$ for $z \in D_i \cap D_{i+1}$

Singularities of an Analytic [redacted] Function

[redacted] The analytic function f, defined on region $\mathcal{R}$, may behave "nicely" or "badly" when we go towards points in the boundary $\partial\mathcal{R} = \bar{\mathcal{R}} - \mathcal{R}$ ($\bar{\mathcal{R}}$ is the closure of $\mathcal{R}$).

Examples: a) $f(z) = \frac{1}{z}$ on $\mathbb{C} - \{0\}$, b) $f(z) = \exp(\frac{1}{z^2})$ on $\mathbb{C} - \{0\}$

c) $f(z) = \sqrt{z}$ on $\mathbb{C} - \{x+i0 \mid x \in \mathbb{R}, x \leq 0\}$

d) $f(z) = \ln z$ on $\mathbb{C} - \{x+i0 \mid x \in \mathbb{R}, x \leq 0\}$

The first two examples would have analytic func in neighborhood of 0 except at zero.

Def. The point z_0 is an isolated singularity of f if f is analytic at every point in some neighborhood of z_0 except a z_0.

Note that, for example (a), $z f(z)$ ~~is~~ can be made analytic at 0.

Def! An isolated singularity at z_0 is a pole if, for some positive integer n, $(z-z_0)^m f(z) = g(z)$ where g in an analytic function defined on [redacted] neighborhood of z_0 (including z_0). The order of the pole is the smallest n for which this is true.

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Def: Isolated [redacted] singularities [redacted] which are not a pole[redacted] is an essential singularity

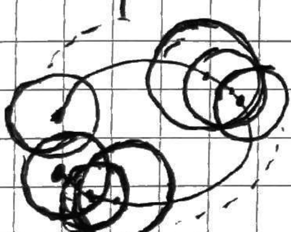
Example b) is like that.

Now we come to the messy case of branch points, [redacted] exemplified by c) and d)

One way to think about these points is to consider analytic continuation around it.

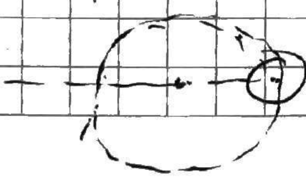
Consider a curve $\gamma : [a, b] \to \mathbb{C}$. Let $t_i \in [a, b]$ for $i = 0, \ldots, n$, with $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$

The points $\gamma(t_i)$ are at the centers of open disc D_i with the $\gamma([t_i, t_{i+1}]) \subset D_i$. This known as a chain of discs covering a parametrized curve. An analytic continuation on these discs is an analytic cont. along γ.



Let $(f_0, D_0), \ldots, (f_n, D_n)$ be an analitic continuation of (f_0, D_0) along γ. If $D_n \cap D_0 \neq \phi$, there is no guarantee that $f_0(z) = f_n(z)$ for $z \in D_0 \cap D_n$

Consider Example c). Consider a closed path $\gamma : [0, 2\pi] \to \mathbb{C}$
$\gamma(\theta) = e^{i\theta}$



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As we go around, $f_0(z) = -f_n(z)$

Example d) $f_0(z) = f_n(z) - 2\pi i$

Def The [redacted] point z_0 is a branch point if, in each neighborhood of z_0, there is a circle [redacted] $K_\rho = \{z = \rho e^{i\theta} \mid \theta \in [0, 2\pi]\}$

Such that: i) the analytic functi[redacted]n f is defined on part of K_ρ

[redacted] ii.) The analytic conti[redacted]nuation of f around K_ρ leads to functional elements that have different values on the same point on K_ρ.

The alternative approach is the realize that the branch point is indicating the the function is defined on a "covering space" of $\mathbb{C}$, namely a Riemann Surface, a multisheet version of $\mathbb{C}$.

[redacted]

Branch cuts are curves C so that $\mathbb{C} - C$ can have an analytic f defined

For example  for $f(z) = \sqrt{z}$
or $f(z) = \ln z$

Branch cuts could be altered for convenience.

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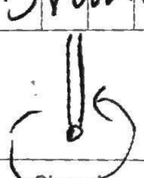


Global Properties of Analytic Functions

__Def__ f is an entire function if it is analytic in all of $\mathbb{C}$.

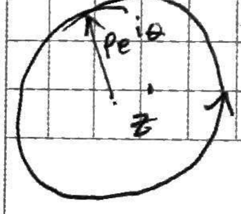
Examples: Polynomials, exponential function.

__Liouville's Thm__: If f is an entire function with its modulus bounded everywhere, then it is a constant.

Comment: The book proves it by talking about its properties around infinity. We will do something more pedestrian.

__Proof__:
$$f'(z) = \frac{1}{2\pi i}\oint_C \frac{f(\xi)}{(\xi - z)^2}\, d\xi$$

where $C = \{\rho e^{i\theta} \mid \theta \in [0, 2\pi]\}$ with $|z| < \rho$.



Let $|f(z)| \le M$ for all $z \in \mathbb{C}$.

Then $|f'(z)| = \frac{1}{2\pi}\left|\oint_C \frac{f(\xi)}{(\xi - z)^2} d\xi\right| \le \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{f(\xi)}{(\xi - z)^2}\right| \rho\, d\theta \le \frac{M\rho}{(\rho - |z|)^2}$

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Read and Understood By

Let $\rho \to \infty$. Then we have $f'(z) = 0$ for all z. We can integrate it back to show $f(z) = $ constant. QED.

Liouville's theorem leads to simple proof of the fundamental theorem of algebra.

<u>Thm</u>: Every non constant polynomial with complex coefficients has at least one complex root.

<u>Proof</u> Let p be a non constant polynomial s.t. $p(z) \neq 0$ [redacted] for any $z \in \mathbb{C}$.
Since it is a continuous function, $1/|p|$ is bounded in any compact region, say $\{z \mid |z| \leq \rho\}$. Also since $1/p(z) \to 0$ as $z \to +\infty$, for sufficiently large ρ, $1/|p|$ is bounded for $\{z \mid |z| \geq \rho\}$.

So [redacted] $1/p$ is bounded everywhere.

[redacted] Hence $1/p$ is a constant $\Rightarrow$ p is a constant.

Contradiction? $\boxed{QED}$.

If $p_n(z)$ is of degree n, [redacted] $n > 0$, and has a root z_n, then $p(z) = (z - z_n)\, p_{n-1}(z)$ where $p_{n-1}(z)$ is polynomial of degree $n-1$. Continuing this process

$$p_n(z) = \alpha (z - z_1) \cdots (z - z_n), \quad \text{[redacted]} \ \alpha \neq 0$$

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