Functions of a Complex Variable

Consider nonempty connected opensets in $\mathbb{C}$. (Connected means this set cannot be written as a union of two nonempty open sets.) We will call such sets [redacted] a region. A closed region [redacted] would be the closure of a region.

Consider a region $R \subset \mathbb{C}$ and a function $f: R \to \mathbb{C}$. This function is differentiable at $z_0 \in R$ if $\lim\limits_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$ exists.

If, in addition, f is differentiable in an open neighborhood Ω of z_0, then f is called analytic (or holomorphic) in Ω.

Example: $f(z) = (\mathrm{Re}\, z)^2 + i$ [redacted] $(\mathrm{Im}\, z)^2$ is differentiable at $z = 0$, but not analytic in the neighborhood of that point:

$z = x + iy$, $z \neq 0$ $\quad \left| \dfrac{x^2 + iy^2}{x + iy} \right| = \sqrt{\dfrac{x^4 + y^4}{x^2 + y^2}}$ $\quad 0 \le x^4 + y^4 =$ [redacted] $(x^2+y^2)^2 - 2x^2y^2 \le (x^2+y^2)^2$

So $\left| \dfrac{x^2 + iy^2}{x + iy} \right| \le \sqrt{x^2 + y^2}$ $\quad$ Hence $\lim\limits_{z \to 0} \dfrac{f(z) - f(0)}{z - 0} = 0$

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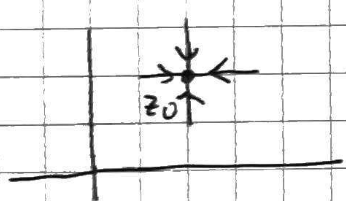
Let $z \in \mathbb{C}$ be written as $x+iy$, $x, y \in \mathbb{R}$, and let there be functions u, v mapping from a subset of $\mathbb{R}^2$ that correspond to $\mathcal{R}$, $\mathcal{R}_2 = \{(x,y) \mid x+iy \in \mathcal{R}\}$, to $\mathbb{R}$ [redacted] s.t. $f : \mathcal{R} \to \mathbb{C}$ could be represented as

$$f(x+iy) = u(x,y) + i\, v(x,y).$$

We could ask what makes f differentiable at $z_0 = x_0 + i y_0$? One necessary condition is that



$$\lim_{\substack{h \in \mathbb{R} \\ h \to 0}} \frac{f(z_0+h) - f(z_0)}{h} = \lim_{\substack{h \in \mathbb{R} \\ h \to 0}} \frac{f(z_0+ih) - f(z_0)}{ih}$$

$$\Rightarrow f'(z_0) = \left[\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}\right]_{(x_0,y_0)} = \frac{1}{i}\left[\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y}\right]_{x_0,y_0}$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

Cauchy-Riemann conditions.

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It turns out that if u, v are functions with continuous first partial derivatives and satisfy Cauchy-Riemann conditions in some neighborhood of [redacted] $z_0 = x_0 + iy_0$, then f is [redacted] differentiable at z_0.

Note that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

implies $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right)$

If the second derivatives of u, v are continuous

then $\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$ and $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Similarly $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$

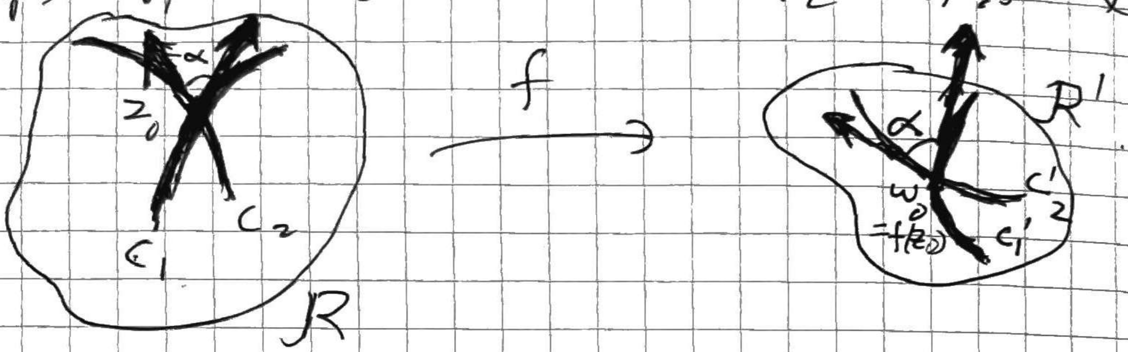
Hence u, v are harmonic functions, Solution of Laplaces equation in 2-dim.

In addition $\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} = 0$, the gradient of u, v are perpendicular

The u, v level sets cut each other at right angles.

Let [redacted] f is an analytical function on $\mathcal{R}$, and let $f : \mathcal{R} \to \mathcal{R}' = f(\mathcal{R})$ be invertible

Consider two smooth curves C_1, C_2 intersecting in $\mathcal{R}$ at z_0. These curves are images of componentwise differentiable [redacted] maps $\gamma_1 : [a_1, b_1] \to \mathbb{C}$ and $\gamma_2 : [a_2, b_2] \to \mathbb{C}$



f maps z_0 to $w_0 = f(z_0)$ and curves C_1, C_2 to C_1', C_2'. If the angle between the tangents of C_1, C_2 at z_0 was α, the angle between tangents at w_0 between C_1', C_2' would also be α.

For smooth curves, let $\gamma(t) = x(t) + i\,y(t)$,

the tangent $\dfrac{d\gamma(t)}{dt} = \dfrac{dx(t)}{dt} + i\,\dfrac{dy(t)}{dt}$,

after application of f

goes to $\frac{d}{dt} f\circ\gamma(t) = \underbrace{f'(\gamma(t))}\frac{d\gamma(t)}{dt}$

This is just multiplication by a complex number!

If two curves pass through the same point z_0 and $\gamma_1(t_1) = z_0$, $\gamma_2(t_2) = z_0$,

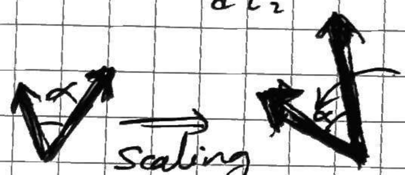
$\frac{d}{dt} f\circ\gamma_1(t_1) = f'(z_0)\frac{d\gamma_1}{dt_1}$ [redacted]

$\frac{d}{dt} f\circ\gamma_2(t_2) = f'(z_0)\frac{d\gamma_2}{dt_2}$

$\Bigg\}$ $Ae^{i\theta}\frac{d\gamma}{dt}$

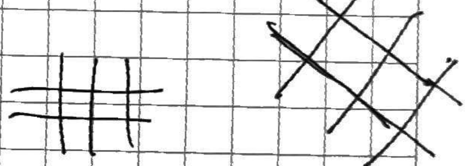
If $f'(z_0) = Ae^{i\theta}$, A positive

[redacted]

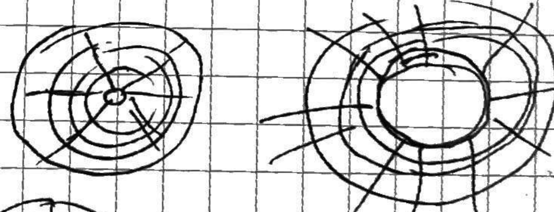


rotation by the same angle θ.

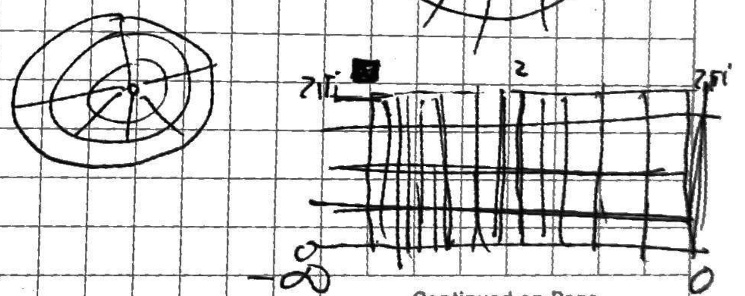
Examples: a) $z \to az+b$



b) $0<|z|<1$ $\quad z \to \frac{1}{z}$



c) $0<|z|<1$ $\quad z \to \log z$



d) $\mathrm{Re}\, z>0$ $\quad z \to \sqrt{z}$

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Read and Understood By

These methods often help in solving 2d potential and other problems.

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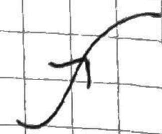
Integration in the Complex Plane

We want to define $\int_C f(z)\,dz$

on a curve with an orientation.



C is related to a smooth map $t \longmapsto x(t)+iy(t)$

for $t \in [a,b]$

If $f(z) = u(x,y) + iv(x,y)$

We could define

$$\int_a^b (u+iv)\,(x'(t)+iy'(t))\,dt$$

as $\int_a^b \left[u(x(t),y(t))$ [redacted] $x'(t) - v(x(t),y(t))\,y'(t)\right] dt$

$+\, i \int_a^b \left[u(x(t),y(t))\,y'(t) + v(x(t),y(t))\,x'(t)\right] dx$

[redacted] $= P + iQ$

Notice that $P = \int (u\,dx - v\,dy)$

and $Q = \int($ [redacted] $+ u\,dy)$

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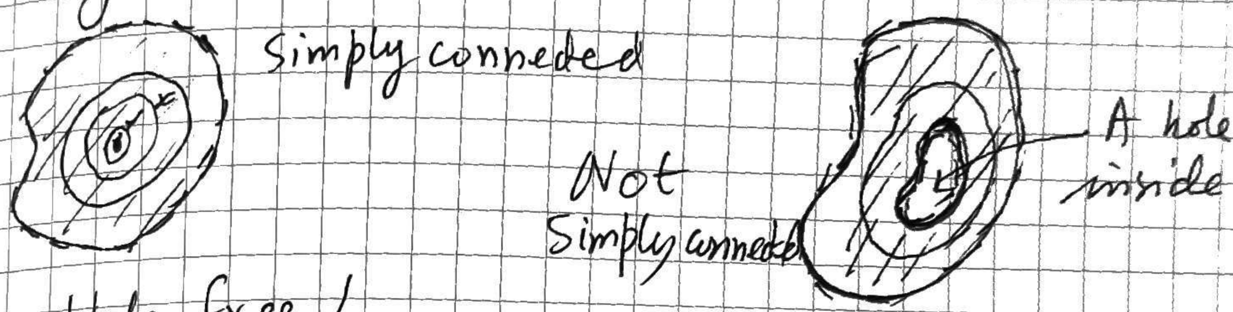
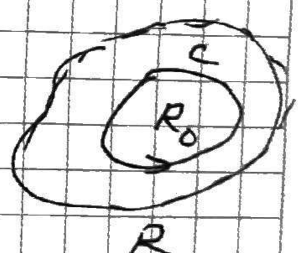
look like integrals of on forms.

Imagine now that we have a simply connected region R. A careful definition of simply connected require definition of homotopy: [redacted] Informally, it means any closed loop could be deformed to a point continuously.



Hole free!

Let us now consider a closed region R_0 inside a simply connected R which can be thought of as a compact manifold with a boundary, and a closed curve $C = \partial R$.



$$\oint_C f(z)\,dz = \int_{\partial R_0} (u\,dx - v\,dy) + i\int_{\partial R_0} (v\,dx + u\,dy)$$

By Stokes theorem this is $\int_{R_0} d(u\,dx - v\,dy) + i\int_{R_0} d(v\,dx + u\,dy)$

[redacted]

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$$\oint f dz = \int_{\text{[redacted]} R_0} \left(-\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} \right) dx \wedge dy$$

$$+ i \int_{R_0} \left(-\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x} \right) dx \wedge dy$$

$= 0$ by Cauchy-Riemann conditions when f is analytic in R.

This is one of the many avatars of Cauchy's theorem.

<u>Cauchy's Theorem</u>: If f is analytic in a simply-connected region R, and f' is continuous in the region, then for a closed contour C,

$$\boxed{\oint_C f(z)\, dz = 0.}$$

We have not precisely defined closed contours, because many many definitions work.

By the way, it is sometimes useful to have $z, \bar{z}$ as the coordinates.

$z = x + iy, \quad \bar{z} = x - iy \Big\} \Rightarrow x = \frac{z+\bar{z}}{2}$
$y = \frac{z-\bar{z}}{2i}$

Formally
$$\frac{\partial}{\partial z} = \frac{\partial x}{\partial z}\frac{\partial}{\partial x} + \frac{\partial y}{\partial z}\frac{\partial}{\partial y}$$
$$= \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$
$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

Cauchy-Riemann condition can be written as $\frac{\partial}{\partial \bar{z}} f(z) = 0$

We can similarly define differential forms $dz = dx + i\,dy \qquad d\bar{z} = dx - i\,dy$

We can consider the regions R as complex manifold with complex tangent spaces an cotangent spaces.

So $df(z) = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z} = f'(z)\,dz$

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