The Metric Tensor

<u>Def</u>: A smooth n-dimensional manifold M endowed with a smooth $(0,2)$ nondegenerate symmetric tensor field g is called a [redacted] pseudo-Riemannian manifold. The tensor field g is called the metric tensor

At any point $p \in M$ if $u, v \in T_pM$ we can compute a number $g(u,v)$ which is a scalar product. In components, it looks like $g_{jk} u^j v^k$. Note that

$v \mapsto g_p(\bullet, v) = \tilde{v}$ is a map from T_pM to T_p^*M
[redacted] In components v^j [redacted] $\to \tilde{v}_j = g_{jk} v^k$ (index lowering)
Nondegenerate g means this map is invertible

We have $\bar{g}_p : T_p^*M \to T_pM$

$\bar{g}^{jk} g_{kl} = \delta^j_l$, $\quad \omega \to \bar{g}(\bullet, \omega)$ in components $\tilde{\omega}^j$ [redacted] $= \bar{g}^{jk} \omega_k$ (index raising)

g is a symmetric 'matrix' with nonzero real eigenvalues. If it has p +ve and q -ve eigenvalues, then signature of g is (p,q)

Metric Tensor and Distance

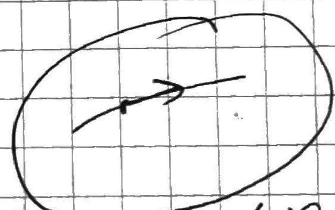
In Relativity theory, we deal with signatures like (1, 3). However, in this section, we focus on (n, 0) signature!

Def: If $g(u,u) \geq 0$ and $g(u,u)=0$ implies $u=0$, then g is positive definite metric. A [redacted] smooth manifold with a smooth positive definite metric is a smooth Riemannian manifold.

Now imagine you have smooth curve on the manifold

$\gamma : I \to M$

$I=[a,b]$



At every point on the curve, we have a tangent vector. [redacted] At point $\gamma(\lambda)$, for $\lambda \in I$, [redacted] there is a vector $v(\lambda) \in T_{\gamma(\lambda)} M$. For this curve,

$$l(\gamma) = \int_a^b \sqrt{g(v(\lambda), v(\lambda))}\, d\lambda$$ (remember $g(v(\lambda), v(\lambda)) \geq 0$)

defined its length. If we describe [redacted] the curve by $x^i = x^i(\lambda)$,

$$l(\gamma) = \int_a^b \sqrt{g_{jk} \frac{dx^j}{d\lambda} \frac{dx^k}{d\lambda}}\, d\lambda$$

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PROJECT

In textbooks talking about infinitesimal arclengths, we say

$ds^2 = g_{jk}\, dx^j dx^k$.

This is a rather useful tool for building intuition and we will gladly use it

We could define $s(\lambda) = \int_a^\lambda \sqrt{g(\dot{x}(\lambda'), \dot{x}(\lambda'))}\, d\lambda'$ along the curve and show

$$\left(\frac{ds}{d\lambda}\right)^2 = \left(\frac{d\, s(\lambda)}{d\lambda}\right)^2 = g_{jk} \frac{dx^j}{d\lambda} \frac{dx^k}{d\lambda}$$, which is a rigorous statement. This statement depends on a parametrization, though.

Example: 1) Two-dimensional plane, $\mathbb{R}^2$

$ds^2 = dx^2 + dy^2 \leftarrow$ Cartesian

$\quad = dr^2 + r^2 d\theta^2 \leftarrow$ Polar

2) $\mathbb{R}^3$: $\quad ds^2 = dx^2 + dy^2 + dz^2$

$\quad\quad = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$

3) S^2: $\quad ds^2 =$ [redacted] $d\theta^2 + \sin^2\theta\, d\phi^2$

Unit sphere

Metric [redacted] Tensor and a Volume form.

When we change coordinates

g_{jk} tranforms to $g'_{mn} = \frac{\partial x^j}{\partial x'^m} \frac{\partial x^k}{\partial x'^n} g_{jk}$

$$= \left[(J^{-1})^T \underset{\uparrow \text{matrix}}{g} J^{-1} \right]_{mn}$$

$J^n{}_k = \frac{\partial x'^n}{\partial x^k}$

So $\det(\underset{\uparrow \text{matrix}}{g'}) = \det(\underline{g}) \left[\det(\underline{J}) \right]^{-2}$

Note that $dx'^1 \wedge \cdots \wedge dx'^n = \frac{\partial x'^1}{\partial x^{i_1}} \cdots \frac{\partial x'^n}{\partial x^{i_n}} dx^{i_1} \wedge \cdots \wedge dx^{i_n}$

$= \varepsilon^{i_1 \cdots i_n} \frac{\partial x'^1}{\partial x^{i_1}} \cdots \frac{\partial x'^n}{\partial x^{i_n}} dx^1 \wedge \cdots \wedge dx^n$

$= \det(\underline{J}) \, dx^1 \wedge \cdots \wedge dx^n$

So $\Omega = \sqrt{\det(g)} \, dx^1 \wedge \cdots \wedge dx^n$ is invariant under tranformation.

For pseudo Riemannian metrics one often defines $\Omega = \sqrt{|\det(g)|} \, dx^1 \wedge \cdots \wedge dx^n$.
(Since g is nondeg and fixed signature, therefore, $\det(g)$ has the same sign everywhere in a connected part of M)

Ω allows us to define Hodge duals

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PROJECT

The Laplacian operator

$$\Omega(x) = \rho(x)\, dx^1 \wedge \cdots \wedge dx^n$$

$$\rho(x) = \sqrt{|\det(g)|}$$

Now consider a contravariant vector associated with gradient of a function: $\delta f = \bar{g}(\cdot, df)$

In components, $(\delta f)^j = g^{jk} \dfrac{\partial f}{\partial x^k}$

$*\delta f \in \Lambda^{n-1}($[redacted]$M)$, [redacted] $d*\delta f \in \Lambda^n(M)$

$d*\delta f$ is an n-form. $*d*\delta f$ is back to $\Lambda^0(M)$ of a smooth function. If you chase through indices

$$*d*\delta f = \frac{1}{(n-1)!}\varepsilon^{i_1 \cdots i_{n-1} m} \frac{1}{\rho}\,[\text{redacted}]\, \frac{\partial}{\partial x^m} \varepsilon_{i_1 \cdots i_{n-1} j}\, \rho\, g^{jk} \frac{\partial f}{\partial x^k}$$

$$= \delta^m_j \frac{1}{\rho}\frac{\partial}{\partial x^m} \rho\, g^{jk}\frac{\partial f}{\partial x^k}$$

$$= \frac{1}{\sqrt{|\det(g)|}} \frac{\partial}{\partial x^j} \sqrt{|\det(g)|}\, g^{jk} \frac{\partial f}{\partial x^k}$$

So we define, [redacted] $\Delta f = *d*\delta f$ to be the Laplacian

Example: $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$ $\quad \underline{g} = \begin{pmatrix} \ & & \\ & r^2 & \\ & & r^2\sin^2\theta \end{pmatrix}$

$\det(g) = r^4 \sin^2\theta$. $\quad \sqrt{\det(g)} = r^2 \sin\theta$

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$$\Delta = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r} r^2 \sin\theta \frac{\partial}{\partial r} + \frac{\partial}{\partial \theta} r^2 \sin\theta \frac{1}{r^2}\frac{\partial}{\partial \theta} + \frac{\partial}{\partial \phi} r^2 \sin\theta \frac{1}{r^2 \sin^2\theta}\frac{\partial}{\partial \phi}\right]$$

Since $\bar{\underline{g}} = \begin{pmatrix} 1 & & \\ & \frac{1}{r^2} & \\ & & \frac{1}{r^2\sin^2\theta} \end{pmatrix}$

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin\text{[redacted]}\theta}\frac{\partial}{\partial \theta}\sin\theta\frac{\partial}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}$$

Of course, in the cartesian system

$\underline{g} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ $\bar{\underline{g}} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$ So

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

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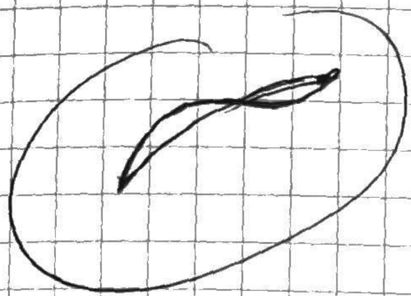
Geodesic curve on a manifold



How to find the shortest path.

$$l = \int_a^b \sqrt{g_{jk}(x)\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda}}\, d\lambda = \int_a^b \sqrt{\sigma}\, d\lambda$$

$$x^i(\lambda) \to x^i(\lambda) + \epsilon\eta^i(\lambda) \qquad \eta^i(a) = \eta^i(b) = 0$$

$$\frac{dl(\epsilon)}{d\epsilon} = \frac{1}{2}\int_a^b \frac{1}{\sqrt{\sigma}}\left[\frac{\partial g_{jk}}{\partial x^m}\eta^m \frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda} + g_{jk}\frac{d\eta^j}{d\lambda}\frac{dx^k}{d\lambda} + g_{jk}\frac{dx^j}{d\lambda}\frac{d\eta^k}{d\lambda}\right] d\lambda$$

$$= \frac{1}{2}\int_a^b \frac{1}{\sqrt{\sigma}}\left[\frac{\partial g_{jk}}{\partial x^m}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda}\eta^m + 2g_{jm}\frac{dx^j}{d\lambda}\frac{d\eta^m}{d\lambda}\right] d\lambda$$

$$= \int_a^b \left[\frac{1}{2}\frac{1}{\sqrt{\sigma}}\frac{\partial g_{jk}}{\partial x^m}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda} - \frac{d}{d\lambda}\frac{1}{\sqrt{\sigma}} g_{jm}\frac{dx^j}{d\lambda}\right]\eta^m\, d\lambda$$

$$= -\int_a^b \left[\frac{1}{\sqrt{\sigma}}\frac{d}{d\lambda} g_{jm}\frac{1}{\sqrt{\sigma}}\frac{dx^j}{d\lambda} - \frac{1}{2}\frac{\partial g_{jk}}{\partial x^m}\frac{1}{(\sqrt{\sigma})^2}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda}\right]\eta^m\sqrt{\sigma}\, d\lambda$$

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We can plug in [redacted] $\sigma = g_{jk}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda}$ and push on (the book does). We should realize $l(\gamma)$ does not depend on the parametrization

If we go from a solution $x^i(\lambda)$ to $x^i(\lambda(\mu)) = \tilde{x}^i(\mu)$, this will [redacted] also be a solution. To fix this degeneracy, we define $\frac{ds}{d\lambda}$ to be $\sqrt{\sigma}$ [redacted].

So s is the arclength.

$$\frac{dl}{d\varepsilon} = -\int \left[\frac{d}{ds} g_{jm}\frac{dx^j}{ds} - \frac{1}{2}\frac{\partial g_{jk}}{\partial x^m}\frac{dx^j}{ds}\frac{dx^k}{ds}\right]\eta^m \, ds$$

Since $\eta^m(s)$ is arbitrary,

$$\frac{d}{ds}\left(g_{mj}\frac{dx^j}{ds}\right) - \frac{1}{2}\frac{\partial g_{jk}}{\partial x^m}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0$$

$$g_{mj}\frac{d^2x^j}{ds^2} + \left(\frac{\partial g_{mj}}{\partial x^k} - \frac{1}{2}\frac{g_{jk}}{\partial x^m}\right)\frac{dx^j}{ds}\frac{\partial x^k}{ds} = 0$$

$$g_{mj}\frac{d^2x^j}{ds^2} + \Gamma_{m,jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0$$

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$\Gamma_{m,jk}$ are christoffel symbols ([redacted] of the 1st kind).

$$\Gamma_{m,jk} = \frac{1}{2}\left(\frac{\partial g_{mj}}{\partial x^k} + \frac{\partial g_{mk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m}\right)$$

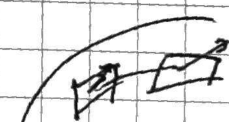
Use g^{im} to raise indices with

$$\Gamma^i_{jk} = g^{im}\,\Gamma_{m,jk}$$ (Christoffel symbols of the second kind)

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0.$$

We have the constr. $g_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 1$ satisfied!

Γ^i_{jk} is associated with connections and parallel transport, allowing [redacted] comparison of vectors in neighboring tangent spaces



Example: S^2: $g_{\theta\theta} = 1$, $g_{\phi\phi} = \sin^2\theta$

Only nontrivial derivative: $\frac{\partial g_{\phi\phi}}{\partial\theta} = 2\sin\theta\cos\theta$

So $\Gamma_{\theta,\phi\phi} = \frac{1}{2}\left(-\frac{\partial g_{\phi\phi}}{\partial\theta}\right) = -\sin\theta\cos\theta \Rightarrow \Gamma^\theta_{\phi\phi} = g^{\theta\theta}\Gamma_{\theta,\phi\phi} = -\sin\theta\cos\theta$

$\Gamma_{\phi,\theta\phi} = \Gamma_{\phi,\phi\theta} = \frac{1}{2}\frac{\partial g_{\phi\phi}}{\partial\theta} = \sin\theta\cos\theta \Rightarrow \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = g^{\phi\phi}\Gamma_{\phi,\theta\phi}$ [redacted]

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Read and Understood By $= \frac{1}{\sin^2\theta}\sin\theta\cos\theta = \cot\theta$

So the equations are $\frac{d^2\theta}{ds^2} + \Gamma^{\theta}_{\phi\phi}\left(\frac{d\phi}{ds}\right)^2 = 0$, and $\frac{d^2\phi}{ds^2} + 2\Gamma^{\phi}_{\theta\phi}\frac{d\theta}{ds}\frac{d\phi}{ds} = 0$

Or $\ddot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 = 0$ and $\ddot{\phi} + 2\cot\theta\,\dot{\phi}\dot{\theta} = 0$

Note that if we have $\dot{\phi} = 0$, [redacted] initially then $\dot{\theta} = \omega$ is a solution

The solution moves along ϕ = constant lines providing great circles. Since $\left(\frac{d\theta}{ds}\right)^2 = 1$

$\omega = \pm 1$.

For general soln recast second eqn as $\frac{d}{dt}\left(\dot{\phi}\sin^2\theta\right) = 0$

or $\dot{\phi} = \frac{c}{\sin^2\theta}$ ("conserved quantity [redacted]" c like angular mom) $\Rightarrow$ $\ddot{\theta} - \frac{c^2\cos\theta}{\sin^3\theta} = 0$

implying $\frac{1}{2}\dot{\theta}^2 + \frac{c^2}{2\sin^2\theta} = \frac{1}{2}A$ (like energy) is [redacted] conserved!

Note that we have $\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2 = A$, which has to be 1.

So $\dot{\theta}^2 + \frac{c^2}{\sin^2\theta} = 1$ and $\dot{\phi} = \frac{c}{\sin^2\theta}$ $\Rightarrow$ $\frac{d\phi}{d\theta} = \frac{\dot{\phi}}{\dot{\theta}} = \frac{\pm\, c/\sin^2\theta}{\sqrt{1 - c^2/\sin^2\theta}} = \frac{\pm\, c\csc^2\theta}{\sqrt{1 - c^2\csc^2\theta}}$

$$= \frac{\pm\, c}{\sin\theta\sqrt{\sin^2\theta - c^2}}$$

$$\left.\begin{aligned} d(\cot\theta) &= -\csc^2\theta\, d\theta \\ \csc^2\theta &= 1 + \cot^2\theta \end{aligned}\right\}$$ So $u = \cot\theta$ substitution allows us to integrate

$$\int_{\phi_0}^{\phi} d\phi' = \pm\int_{\pi/2}^{\theta} \frac{c\csc^2\theta'\,d\theta'}{\sqrt{1 - c^2\csc^2\theta'}} = \mp\int \frac{c\,du'}{\sqrt{1 - c^2 - c^2u'^2}} = \mp\int \frac{du}{\sqrt{a^2 - u^2}} \qquad a^2 = \frac{1 - c^2}{c^2}$$

$\phi - \phi_0 = \sin^{-1}(u/a)$ $\Rightarrow$ $\cot\theta = a\sin(\phi - \phi_0)$

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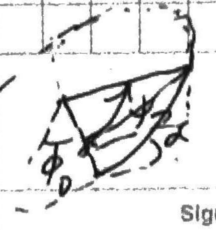

$\cos\theta = \sin\psi\sin\alpha$ $\quad \sin\theta = \sqrt{\sin^2\psi\cos^2\alpha + \cos^2\psi}$

$\sin(\phi - \phi_0) = \frac{\sin\psi\cos\alpha}{\sqrt{\sin^2\psi\cos^2\alpha + \cos^2\psi}}$ $\Rightarrow$ $\cot\theta = \tan\alpha\sin(\phi - \phi_0)$

$a = \tan\alpha$