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Exterior Derivative and Stokes Theorem

Let $\Lambda^p(M)$ be the space of all smooth forms on M. We [redacted] define map $d : \Lambda^p(M) \to \Lambda^{p+1}(M)$ (for every p) by requiring the following properties:

a) For functions $f \in \Lambda^0(M)$
   df is just the [redacted] differential

b) d is a linear map

c) $d(\sigma \wedge \tau) = d\sigma \wedge \tau + (-1)^p \sigma \wedge d\tau$
   when $\sigma \in \Lambda^p(M)$ and $\tau \in \Lambda^q(M)$

d) $d(d\sigma) = 0$ for any form σ.

It turns out that this d is uniquely defined by these properties. It is called the exterior derivative.

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PROJECT

0. We saw $df = \frac{\partial f}{\partial x^i} dx^i$ $(df)_i = (\nabla f)_i$

Let us see a few specific examples of action of the d operator

1. $\sigma = \sigma_k dx^k$ is a one-form.

$$d\sigma = d\sigma_k \wedge dx^k + \sigma_k \overset{\to \text{zero}}{d(dx^k)}$$

$$= \frac{\partial \sigma_k}{\partial x^j} dx^j \wedge dx^k = \frac{1}{2}\left(\frac{\partial \sigma_k}{\partial x^j} - \frac{\partial \sigma_j}{\partial x^k}\right) dx^j \wedge dx^k$$

BTW, if $\sigma = df$ [redacted] $\sigma_k = \frac{\partial f}{\partial x^k}$, note that

[redacted] $d\sigma = \frac{1}{2} \underbrace{\left(\frac{\partial^2 f}{\partial x^j \partial x^k} - \frac{\partial^2 f}{\partial x^k \partial x^j}\right)}_{\text{zero}} dx^j \wedge dx^k = 0$

as it should be! $d(df) = 0$!

The two form components are components of a curl operator [redacted] from vector calculus

$\vec{\nabla} \times \vec{\sigma} = (\partial_y \sigma_z - \partial_z \sigma_y, \partial_z \sigma_x - \partial_x \sigma_z, \partial_x \sigma_y - \partial_y \sigma_x) \Leftrightarrow ((d\sigma)_{yz}, (d\sigma)_{zx}, d\sigma_{xy})$

[redacted]

$(\vec{\nabla} \times \vec{\sigma})_x \leftrightarrow (d\sigma)_{yz}$, $(\vec{\nabla} \times \vec{\sigma})_y \leftrightarrow (d\sigma)_{zx}$, $(\nabla \times \sigma)_z \leftrightarrow (d\sigma)_{xy}$

$$(\vec{\nabla} \times \vec{\sigma})^i = \frac{1}{2} \varepsilon^{ijk} d\sigma_{jk}$$

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2. Take a two form $\sigma = \frac{1}{2}\sigma_{kl}\, dx^k \wedge dx^l$

$$d\sigma = \frac{1}{2} \frac{\partial \sigma_{kl}}{\partial x^m}\, dx^m \wedge dx^k \wedge dx^l$$

$$= \frac{1}{6}\left(\frac{\partial \sigma_{kl}}{\partial x^m} + \frac{\partial \sigma_{mk}}{\partial x^l} + \frac{\partial \sigma_{lm}}{\partial x^k}\right) dx^k \wedge dx^l \wedge dx^m$$

In 3d, the volume form $\Omega = \frac{1}{6}\epsilon^{\text{[redacted]}jkl}\, dx^j \wedge dx^k \wedge dx^l$

$= dx^1 \wedge dx^2 \wedge dx^3$

Let $\sigma_{jk} = \epsilon_{jkm} V^m$ (Hodge dual)

$d\sigma =$ [redacted] $\frac{1}{2}\epsilon_{kln} \frac{\partial V^n}{\partial x^m}\, dx^m \wedge dx^k \wedge dx^l$

If $dx^m \wedge dx^k \wedge dx^l$ is nonzero all indices are distinct, and they have value 1, 2, 3 in [redacted] some order. For ϵ_{kln} to be nonzero, same has to be true of indices k, l, n. If $k=1, l=2, m=3$ and so [redacted] is n. So $m=n$. Same argument applies for other nontrivial choices for k, l

More formally, $d\sigma = \frac{1}{2}\epsilon_{kln}\epsilon^{klm} \frac{\partial V^n}{\partial x^m}\, dx^1 \wedge dx^2 \wedge dx^3$

$= \delta^m_n \frac{\partial V^n}{\partial x^m}$ [redacted] $\Omega = \frac{\partial V^m}{\partial x^m}\Omega = \mathrm{div}(V)\,\Omega$

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$(\mathrm{div}\, V)\,\Omega = d * V$

$(\vec{\nabla}\cdot\vec{V})$ [redacted] comes from d of a two form.

$d^2=0 \qquad \vec{\nabla}\times\vec{\nabla} f = 0 \qquad \vec{\nabla}\cdot(\vec{\nabla}\times\vec{\sigma}) = 0$

Def: If σ is a p form and V is a vector, the contraction $i_V\sigma$ of σ with V is defined by

$$i_V\sigma \equiv \sigma(V,\cdot)$$

In components, $(i_V\sigma)_{i_1\cdots i_{p-1}} = \sigma_{k i_1\cdots i_{p-1}} V^k$.

Thm For a [redacted] one form field σ, and a vector field V

$$\mathcal{L}_V\sigma = d(i_V\sigma) + i_V(d\sigma)$$

$$(\mathcal{L}_V\sigma)_k = V^\ell\frac{\partial\sigma_k}{\partial x^\ell} + \sigma_\ell\frac{\partial V^\ell}{\partial x^k}$$

Proof:

$$d(i_V\sigma)_k = d(V^\ell\sigma_\ell \text{ [redacted] })_k \text{ [redacted]}$$

$$= \frac{\partial V^\ell}{\partial x^k}\sigma_\ell + V^\ell\frac{\partial\sigma_\ell}{\partial x^k} \text{ [redacted]}$$

$$(i_V(d\sigma))_k = V^\ell\left(\frac{\partial\sigma_k}{\partial x^\ell} - \frac{\partial\sigma_\ell}{\partial x^k}\right) = V^\ell\frac{\partial\sigma_k}{\partial x^\ell} - V^\ell\frac{\partial\sigma_\ell}{\partial x^k}$$

Adding the last two, we get

$$\frac{\partial V^\ell}{\partial x^k}\sigma_\ell + V^\ell\frac{\partial\sigma_k}{\partial x^\ell} = (\mathcal{L}_V\sigma)_k$$

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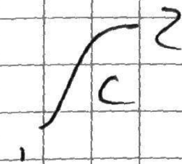
Remark: This relation is true for arbitrary p-forms σ.

Stokes' Theorem and its Generalizations

In vector calculus, we have seen relations like

$$\int_C \nabla f \cdot d\vec{\ell} = f(2) - f(1)$$

(Fundamental Thm of Calc)

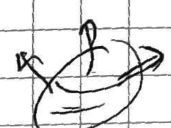


$$\int_S \vec{\nabla} \times \vec{\sigma} \cdot dS = \oint \sigma \cdot d\vec{\ell}$$

(Stokes theorem)

$\int_V \vec{\nabla} \cdot \vec{V}$ [redacted] $\Omega = \oint_S \vec{V} \cdot d\vec{S}$

(Gauss Theorem)



The generalized Stokes' theorem is a relation of the form

$$\int_R d\sigma = \int_{\partial R} \sigma$$

Where σ is a p form (with compact support), R is an oriented submanifold [redacted] of dimension $p+1$, with an oriented boundary ∂R of dimension p.

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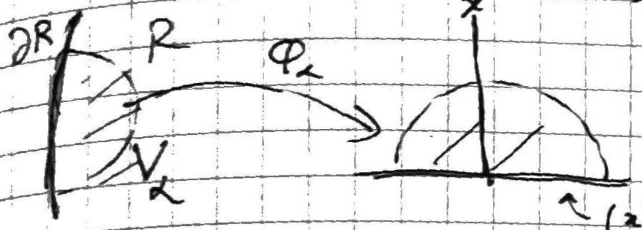
If σ ~~has~~ compact support, $\mathrm{supp}(\sigma)$ is covered by finitely many $\mathrm{supp}(\rho_\alpha)$. We can write $\sigma = \sum_\alpha \sigma \rho_\alpha$, with partitions of unity, with a finite sum over α, being relevant.

We will briefly sketch a proof, to give some intuition about how it works. Use partitions of unity ρ_α

$\int_R d\sigma = \sum_\alpha \int_R d(\rho_\alpha \sigma)$. We want to show this is just $\sum_\alpha \int_{\partial R} (\rho_\alpha \sigma)$

Near the boundary, we have charts mapping to $\mathbb{H}^n$, with $x^n \geq 0$. The boundary is specified by $(x^1, x^2, \cdots, x^{n-1}, 0)$. If there is an bump function ρ_α whose support is a subset of V_α, we could try integrals $\int_{V_\alpha} d(\rho_\alpha \sigma)$



In coordinates, this looks like

$$\frac{1}{p!} \int \partial_k (\rho_\alpha \sigma_{i_1 \cdots i_p})\, dx^k \wedge dx^{i_1} \wedge \cdots \wedge x^{i_p}$$

Since $\rho_\alpha \sigma_{i_1 \cdots i_p}$ vanishes past the support we could do whole halfspace integration $\int_{\mathbb{H}^n} \frac{1}{p!} \partial_k (\rho_\alpha \sigma_{i_1 \cdots i_p})\, dx^k \wedge dx^{i_1} \wedge dx^{i_p}$

Any derivative in $x^1, \cdots, x^{n-1}$ can be integrated out to infinity and we get a zero.

The only one contributing is $\frac{1}{p!} \int_{\mathbb{H}^n} \partial_{x_n} (\rho_\alpha \sigma_{i_1 \cdots i_p})\, dx^n \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}$

The x_n integral gives us $- \int_{x_n = 0} (\rho_\alpha \sigma_{1 \cdots p})\, dx^1 \wedge \cdots \wedge dx^p$

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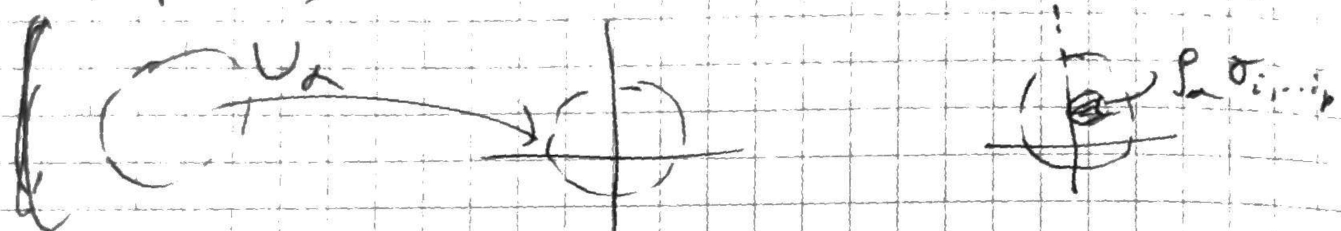
This a contribution from $\int_{\partial R} \sigma$

[redacted] For integrals [redacted] involving bumps f_α completely interior, the integral is zero.



The net contribution from all the bumps intersecting with the boundary gives us

$$\int_R d\sigma = \int_{\partial R} \sigma .$$

Closed and Exact forms

Def: ω is closed is $d\omega = 0$

Def: ω is exact if $\omega = d\sigma$

Poincare lemma: exact $\Rightarrow$ Closed

[redacted] Does [redacted] closed $\Rightarrow$ exact?

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Depends on topology!

$\omega = e$

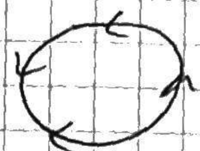
$\omega = dx$

x well defined function

$M = \mathbb{R}$

Closed $\Rightarrow$ exact

$M = S^1$



$\omega = e'$ Locally

$\omega \neq dx$ with a well-defined x.

closed $\not\Rightarrow$ exact.

Related to gauge fields with non-trivial flux passing through the ring.

Other examples: Magnetic monopoles on S^2 etc.

Let us finish by making a few comments on Lagrangian and Hamiltonian dynamics.

In Lagrangian dynamics, one deals with coordinates q^i and velocities $\dot{q}^i$, naturally

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dealing with TM where M is an n-dim manifold. $L(q, \dot{q})$ is a function defined on TM. Hamiltonian dynamics, on the other hand deals with coordinates q^i and momenta $p_i = \frac{\partial L}{\partial \dot{q}^i}$ making p_i naturally coordinates of T^*[redacted]M fibers.

The variational formulation involves $\int_c (p_i dq^i - H dt)$
$p_i dq^i$ is the Poincaré one-form.

$$\delta I = \oint (p_i dq^i - H dt)$$
$$= \int_S d(p_i dq^i - H dt)$$
$$= \int_S (dp_i \wedge dq^i - dH \wedge dt)$$

If X is flow in the (q, p) [redacted] space, (q, p, t) has the flow $(X, \underset{\frac{dt}{dt}}{1})$, we need contraction of [redacted] $dp_i \wedge dq^i - dH \wedge dt$ and the $(X, 1)$ vector to be zero. Call $dp_i \wedge dq^i = \omega$, a closed nondegenerate form, called a symplectic form

[redacted] $\quad i_X \omega = -dH$

$$\omega \rightarrow \begin{array}{c|cc} & q & p \\ \hline q & & -I \\ p & I & \end{array} \qquad (\dot{q}\ \dot{p})\begin{pmatrix} & -I \\ +I & \end{pmatrix} = (\dot{p}, -\dot{q})$$

$$dH \rightarrow \left(\frac{\partial H}{\partial q}, \frac{\partial H}{\partial p}\right)$$

So $\dot{p}_i = -\frac{\partial H}{\partial q^i} \quad \dot{q}_i = \frac{\partial H}{\partial p^i}$