Wedge Product, Exterior/Grassmann Algebra

Consider tensors in $\underbrace{V \otimes \cdots \otimes V}_{p \text{ times}}$. [redacted]

They can also be thought as multilinear maps from $\underbrace{V^* \times \cdots \times V^*}_{p \text{ times}}$ to $\mathbb{R}$.

$$[\text{redacted}](u_1, \dots, u_p) \longmapsto T(u_1, \dots, u_p)$$

We can define subspaces of $V \otimes \cdots \otimes V = \otimes^p V$ by symmetry. Say $\pi \in S_p$ be any permutation.

$$T(u_{\pi(1)}, \dots, u_{\pi(p)}) = T(u_1, \dots, u_p)$$ [redacted]

defines symmetric tensors,

while

$$T(u_{\pi(1)}, \dots, u_{\pi(p)}) = [\text{redacted}]\ \operatorname{sgn}(\pi)\, T(u_1, \dots, u_p)$$

defines the antisymmetric tensors.

The second subspace, [redacted] consisting of antisymmetric tensors, is indicated as $\Lambda^p V$. Notice that it is $\binom{n}{p}$ dimensional space where $n = \dim V$.

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Although [redacted] this construction in general, in the context of differential [redacted] geometry, V would be $T_p^* M$. We will essentially focus on antisymmetric covariant tensors.

Let us start with two 1 forms σ, τ
Consider these two combinations of their tensor products -

$$\omega^S = \frac{1}{2}(\sigma \otimes \tau + \tau \otimes \sigma) \rightarrow \text{Symmetric}$$

$$\omega^A = \frac{1}{2}(\sigma \otimes \tau - \tau \otimes \sigma) \rightarrow \text{Antisymmetric}$$

[redacted] ω^A is an example of the wedge product

$$\sigma \wedge \tau = \frac{1}{2}(\sigma \otimes \tau - \tau \otimes \sigma)$$

We define $\sigma_1 \wedge \cdots \wedge \sigma_p = \frac{1}{p!} \sum_{\substack{i_1 \cdots i_p \\ \in \{1, \ldots, p\}}} \varepsilon_{i_1 \cdots i_p} \sigma_{i_1} \otimes \cdots \otimes \sigma_{i_p}$

where σ_i are oneforms
and

$$\varepsilon_{i_1 \cdots i_p} = \varepsilon^{i_1 \cdots i_p} = +1 \text{ if } i_1, \ldots i_p \text{ is an even permutation of } 1, \ldots, p$$
$$= -1 \quad \text{" " " " odd " "}$$
$$= 0 \quad \text{if any two indices coincide.}$$

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Let us give this definition a trial run.

What is $\sigma_1 \wedge \sigma_2 \wedge \sigma_3$?

$$(1,2,3) \xrightarrow{\text{even permutation}} (1,2,3),(2,3,1),(3,1,2)$$

$$(1,2,3) \xrightarrow{\text{odd permutations}} (2,1,3),(3,2,1),(1,3,2)$$

So
$$\sigma_1 \wedge \sigma_2 \wedge \sigma_3 = \frac{1}{6}\left[\sigma_1 \otimes \sigma_2 \otimes \sigma_3 + \sigma_2 \otimes \sigma_3 \otimes \sigma_1 + \sigma_3 \otimes \sigma_1 \otimes \sigma_2 - \sigma_2 \otimes \sigma_1 \otimes \sigma_3 - \sigma_3 \otimes \sigma_2 \otimes \sigma_1 - \sigma_1 \otimes \sigma_3 \otimes \sigma_2\right]$$

If μ^i form some basis for the vector space V, the general member of $\Lambda^r V$ can be written as $\sigma = \frac{1}{p!}\sigma_{i_1 \cdots i_p}\, \mu^{i_1} \wedge \cdots \wedge \mu^{i_p}$

(we have summation convention, remember!)

If $\sigma \in \Lambda^p V$ and $\tau \in \Lambda^q V$

$$\sigma \wedge \tau = \frac{1}{p!}\frac{1}{q!}\sigma_{i_1 \cdots i_p}\tau_{j_1 \cdots j_q}\, \mu^{i_1} \wedge \cdots \wedge \mu^{i_p} \wedge \mu^{j_1} \wedge \cdots \wedge \mu^{j_q}$$

We have to check it is consistent with our definition for one form and has associativity.

$$\sigma \wedge \tau = (-1)^{pq}\, \tau \wedge \sigma$$

With the wedge product we have full algebra

$$\Lambda(V) = \bigoplus_{p=0}^{n} \Lambda^p(V),$$ called the exterior algebra or Grassmann algebra.

Notice that $\Lambda^n(V)$ is onedimensional with the basis element $\mu^1 \wedge \cdots \wedge \mu^n$. If we change basis to $\{\lambda^i\}$ with $\mu^i = S^i_j \lambda^j$,

$$\mu^1 \wedge \cdots \wedge \mu^n = S^1_{i_1} \cdots S^n_{i_n}\, \lambda^{i_1} \wedge \cdots \wedge \lambda^{i_n}$$

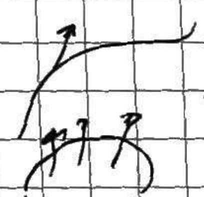
$$= \varepsilon^{i_1 \cdots i_n} S^1_{i_1} \cdots S^n_{i_n}\, \lambda^1 \wedge \cdots \wedge \lambda^n = \det(S)\, \lambda^1 \wedge \cdots \wedge \lambda^n$$

This property would be very important for differential forms and integration.

In vector calculus in 3D, you have seen

line integral $\int_C \vec{A} \cdot d\vec{r}$,

surface integral $\int_S \vec{B} \cdot d\vec{S}$



and volume integral $\int C\, dV$

These are all related to forms.

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Before we move into integrals, let us define one more thing, the Hodge $*$ operator.

Let us say we have a nonzero n-form [redacted] Ω [redacted] in $\Lambda^n(V)$. If $\sigma \in \Lambda^p(V)$ and $\tau \in \Lambda^{n-p}(V)$, $\sigma \wedge \tau \in \Lambda^n(V)$.

Since $\Lambda^n(V)$ is one dimensional,

$$\sigma \wedge \tau = \text{[redacted]}\, c(\sigma, \tau)\, \Omega$$

$\tau \mapsto c(\sigma, \tau)$ is a linear map on $\Lambda^{n-p}(V)$, $*\sigma$, which lives in $\Lambda^{n-p}(V^*)$. If V has an inner product, one could define [redacted] a map from $\Lambda^{n-p}(V^*)$ to $\Lambda^{n-p}(V)$, and $c(\sigma, \tau) = \langle *\sigma, \tau \rangle$.

Let us see that in an example.

Take
$$\left.\begin{aligned} A &= A_1 e^1 + A_2 e^2 + A_3 e^3 \\ B &= B_1 e^1 + B_2 e^2 + B_3 e^3 \end{aligned}\right\} \text{1 forms}$$

$$A \wedge B = (A_1 B_2 - A_2 B_1)\, e^1 \wedge e^2 + (A_2 B_3 - A_3 B_2)\, e^2 \wedge e^3 + (A_3 B_1 - A_1 B_3)\, e^3 \wedge e^1$$

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This is a two form whose components look like it came from the familiar cross product

Now consider $C = C_1 e^1 + C_2 e^2 + C_3 e^3$

$$(A \wedge B) \wedge C = (A_1 B_2 - A_2 B_1)\, C_3\, (e^1 \wedge e^2) \wedge e^3$$
$$+ (A_2 B_3 - A_3 B_2)\, C_1\, (e^2 \wedge e^3) \wedge e^1$$
$$+ (A_3 B_1 - A_1 B_3)\, C_2\, (e^3 \wedge e^1) \wedge e^2$$

$$= \left\{ (A_2 B_3 - A_3 B_2) C_1 + (A_3 B_1 - A_1 B_3) C_2 + (A_1 B_2 - A_2 B_1) C_3 \right\} \underbrace{e_1 \wedge e_2 \wedge e_3}_{\Omega}$$

[redacted]

[redacted] Choose scalar product as the usual dot product

Then $(A_2 A_3 - A_3 B_2)\, e_1$[redacted] $+ (A_3 B_1 - A_1 B_3)\, e_1$[redacted]
$+ (A_1 B_2 - A_2 B_1)\, e_3$[redacted]

is our Hodge $*$ of $A \wedge B$

So, cross product is the Hodge dual in 3d of the two form created by wedge product.

That is why, it is pseudovector. In fact, many pseudovectors you know are infact 2-forms.

In coordinates, if $\Omega(x) = \rho(x) dx^1 \wedge \cdots \wedge dx^n$

$$\sigma \in \Lambda^p(T_p^* M) \to V \in \Lambda^{n-p}(T_p M)$$

$$V^{i_1 \cdots i_{n-p}} = \frac{1}{p!} \frac{1}{\rho(x)} \varepsilon^{i_1 \cdots i_n} \sigma_{i_{n-p+1}, \cdots, i_n}$$

To make a covariant [redacted] tensor out of $V^{i_1 \cdots i_{n-p}}$ we need a metric tensor g_{ij} to lower indices. Such a metric is also associated with a volume form. More about it later!

For completeness sake, from any antisymmetric contravariant $\binom{n-p}{0}$ tensor, we can get a $\binom{0}{n}$ antisymmetric [redacted] covariant tensor.

$$\sigma_{i_1 \cdots i_p} = \frac{1}{(n-p)!} \rho(x) \varepsilon_{i_1 \cdots i_n} V^{i_{p+1} \cdots i_n}$$

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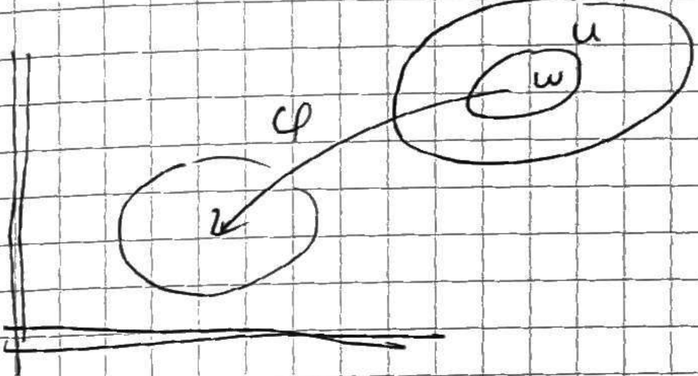
Let us now think of integrals.

Imagine there is an open set $U \subset M$, M being (an n-dimensional) manifold, and [redacted] have an n-form [redacted] field ω defined on U. Let us also have a chart with $\varphi : U \to \mathbb{R}^n$. φ^{-1} is well defined



Consider the pullback $(\varphi^{-1})^* \omega$.

That is $f(x)\, dx^1 \wedge \dots \wedge dx^n$

$$\int_U \omega \equiv \int_{\varphi(U)} (\varphi^{-1})^* \omega = \int_{\varphi(U)} f(x)\, dx^1 \wedge \dots \wedge dx^n$$

thought of as a higher dimensional Riemann integral. Technically we need ω to have compact support, so f has compact support and we can find a rectangle in $\mathbb{R}^n$ which includes the support of f.

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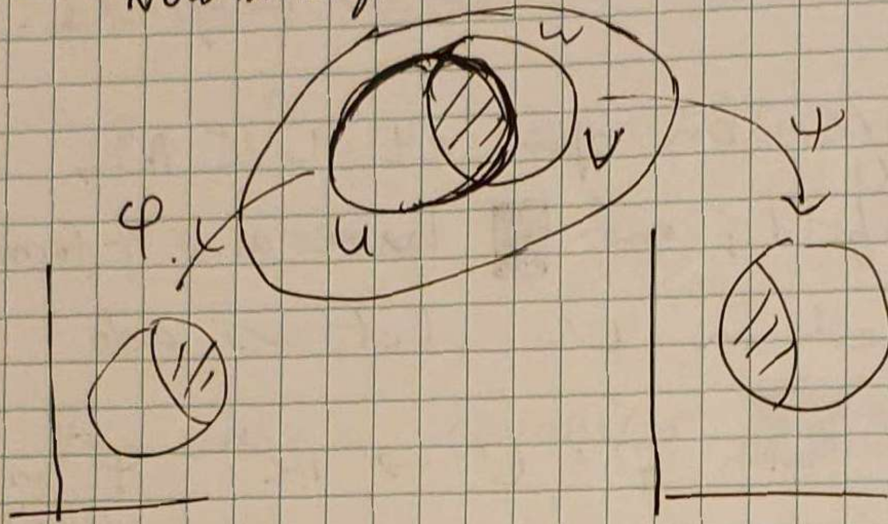
Notice that this is a signed integral $\int f\, dx^1 \wedge dx^2 = -\int f\, dx^2 \wedge dx^1$

For convention we define $\int_{\varphi(U)} f(x)\, dx^1 \wedge \dots \wedge dx^n = \int_{\varphi(U)} f(x)\, |dx^1 \dots dx^n|$

The conventional Riemann integral.

Now imagine we have two charts



$\int_{U\cap V}\omega$ can be defined in two ways $\int_{\varphi(U\cap V)} (\varphi^{-1})^*\omega$ and $\int_{\psi(U\cap V)} (\psi^{-1})^*\omega$

[redacted] Let the corresponding points $(x^1,\dots,x^n)$ and $(x'^1,\dots,x'^n)$.

$f(x)\,dx^1\wedge\dots\wedge dx^n$ transforms to

$$f(x)\frac{\partial x^1}{\partial x'^{i_1}}\cdots\frac{\partial x^n}{\partial x'^{i_n}}\, dx'^{i_1}\wedge\dots\wedge dx'^{i_n}$$

$$= f(x)\det\left\|\frac{\partial x^i}{\partial x'^j}\right\| dx'^1\wedge\dots\wedge dx'^n$$

$$= h(x')\, dx'^1\wedge\dots\wedge dx'^n$$

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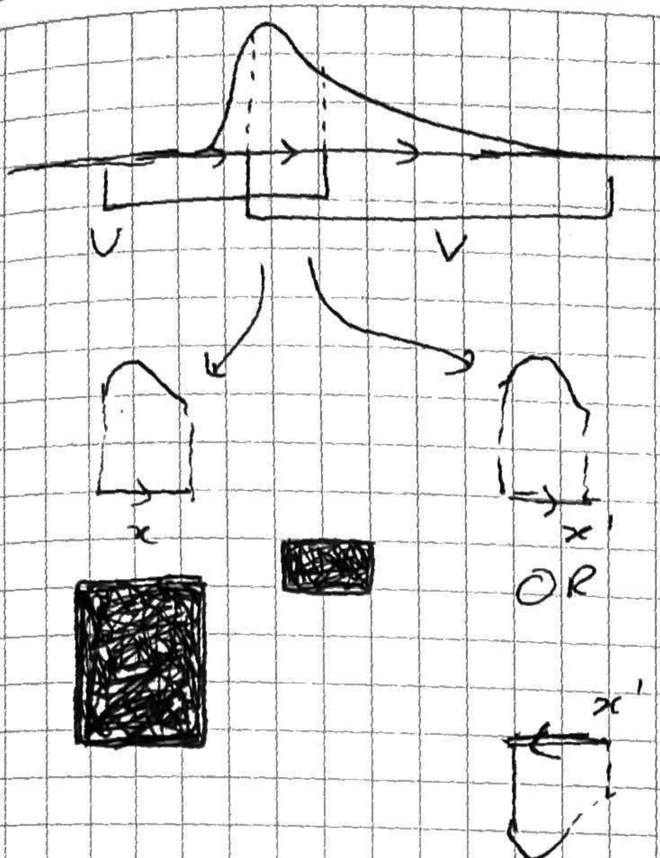
We want $\int f(x)dx^1\cdots dx^n$ and $\int h(x')dx'^1\cdots dx'^n$ to be the same.



[redacted]

[redacted]

[redacted]

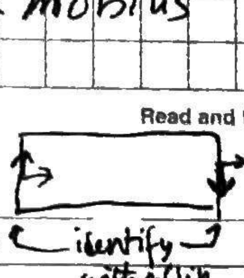
If we do it as an iterated integral, Sometimes going backwards it is ok!

If we define them as [redacted] Riemann integrals with integrations in the increasing directions then we need more.

One solution is to have charts so that $\det\|\frac{\partial x^i}{\partial x'^j}\|$ is always positive! This leads to an oriented atlas.

If it is possible to have such a chart, we will call the manifold orientable!

Cannot be done for manifolds like a Möbius strip.



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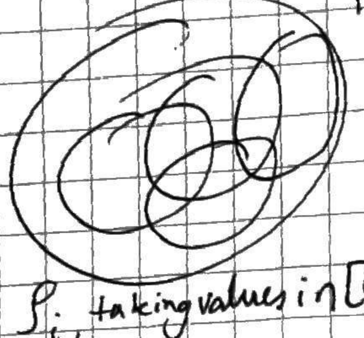
Technically, to define integration on the whole manifolds we need to deal with [redacted] a patchwork of charts $\{U_\alpha, \varphi_\alpha\}$



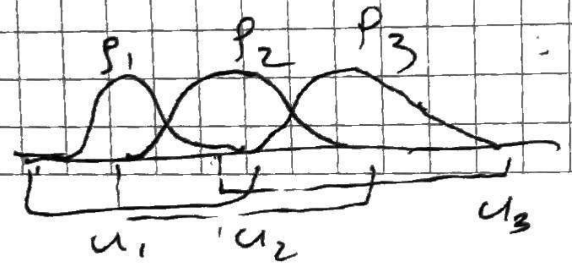
One can show that one can find a countable set of C^∞ functions ρ_i, taking values in $[0,1]$ [redacted], so that each ρ_i has a compact support [redacted] $\mathrm{supp}(\rho_i)$ which is entirely inside on U_α, namely $\mathrm{supp}(\rho_i) \subset U_\alpha$ [redacted] for some α. Make $i \mapsto \alpha(i)$ a function [redacted]. Also at each $p \in M$ only a finite number of ρ_i's are non zero. In addition

$$\sum_i \rho_i(p) = 1 \quad \text{at each } p.$$

These are called a partition of unity.

Then integral $\int_M \omega = \int_M \sum_i \rho_i \omega = \sum_i \int_M \rho_i \omega$

$$= \sum_i \int_{U_{\alpha(i)}} \rho_i \omega$$



Each of these integrals are well-defined.

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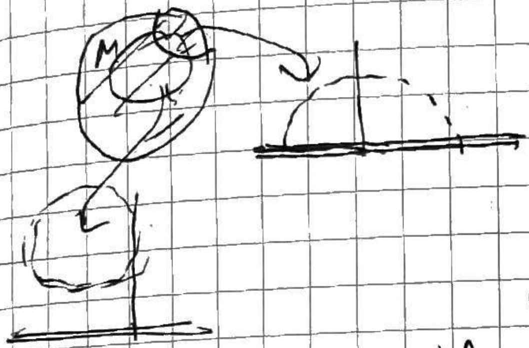
Now, to do integration inside a manifold, we need [redacted] a few more concepts.

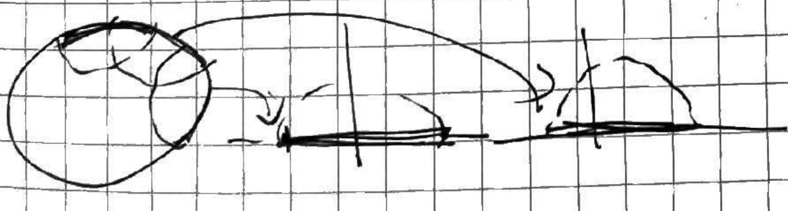
One is the concept of manifolds with boundaries. We need to generalize charts mapping to H^n from $\mathbb{R}^n$



$$H^n = \{(x^1, \ldots, x^n) \in \mathbb{R}^n \mid x^n \geq 0\}$$

Second thing is that the points that map to a point with $x^n = 0$ are on the boundary and have a natural chart with $(x^1, \ldots, x^{n-1}) \Rightarrow$ Induced atlas

Call the manifold of these points ∂M.



Turns out that If M is orientable the induced atlas is oriented.

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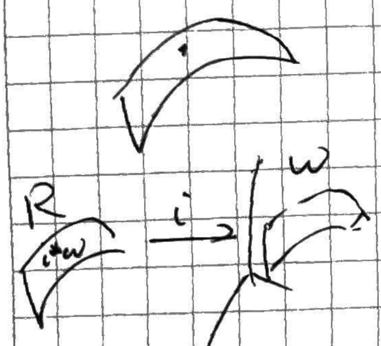
[redacted]

Consider now a submanifold R of M. R could potentially have a boundary ∂R. Let the inclusion map $i : R \to M$ be smooth. For any form field ω on M, there is pullback $i^*\omega$ on R.



If the dimension of R is p and ω is a p-form, with compact support on R, we define

$$\int_R i^*\omega \quad \text{to be} \quad \int_R \omega .$$

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Let us see this in action

Imagine that there is a 2d surface in 3d. (u, v) is the local coordinate system on the surface and $x(u,v), y(u,v), z(u,v)$ provides [redacted] a parametrization of the surface

Let there be a two form $B = B_{yz}(\vec{r})\, dy \wedge dz + B_{zx}(\vec{r})\, dz \wedge dx + B_{xy}(\vec{r})\, dx \wedge dy$

in 3d. In vector calculus, you will write it as $\vec{B} \cdot d\vec{S}$, (B_1, B_2, B_3) being the components of the Hodge dual of [redacted] B.

The integral
$$\int B_{yz} \frac{\partial(y,z)}{\partial(u,v)}\, du\, dv + \int B_{zx} \frac{\partial(z,x)}{\partial(u,v)}\, du\, dv + \int B_{xy} \frac{\partial(x,y)}{\partial(u,v)}\, du\, dv$$

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