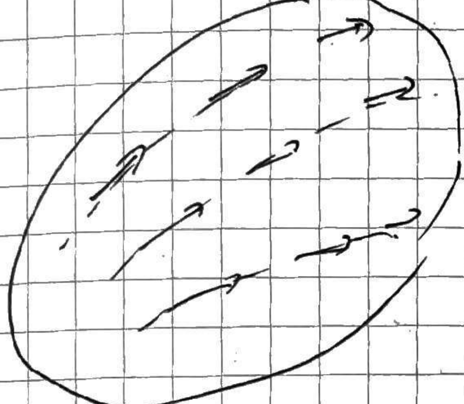
PROJECT

Vector Fields and Integral Curves



Let v be a smooth vector field on a manifold M.

Let $\gamma : I \to M$ be a smooth curve (I being an [redacted] interval in $\mathbb{R}$).

This curve gives rise to a tangent vector u at any point it passes through via the $\frac{d}{d\lambda}$ operator

$$u_{\gamma(\lambda)}(f) = \frac{d}{d\lambda} f(\gamma(\lambda)) \quad \text{for any smooth } f : M \to \mathbb{R}$$

If $\frac{d}{d\lambda} = v_{\gamma(\lambda)}$, then γ is an integral curve of the [redacted] vector field v.

More explicitly, in coordinates,

$$\frac{d}{d\lambda} x^i(\gamma(\lambda)) = v^i(x^1(\gamma(\lambda)), \dots, x^n(\gamma(\lambda)))$$

or $\frac{d}{d\lambda} x^i = v^i(x)$ in short.

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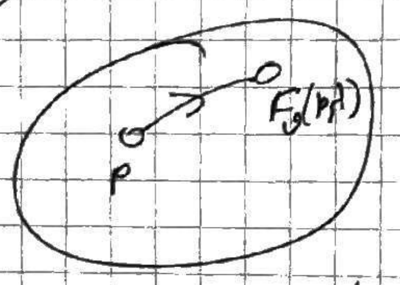
Note that these are a set of differential equations. If $v^i(x)$ are continuous (and Lipshitz continuous), they are going to have a unique local solution.

The map we get by starting out at a point with coordinates $x^i = x^i(0)$ and solve the equations to $x^i(\lambda)$. This solution creates a [redacted] map starting at point p with coordinates $\{x^i\}$ and maps it to a new point that depends on λ and v. We call this map

$$F_v(p,\lambda) : M \to M$$



This turns out to be a smooth invertible map from M to M. To discuss it fully we need to understand smooth maps between smooth manifolds.

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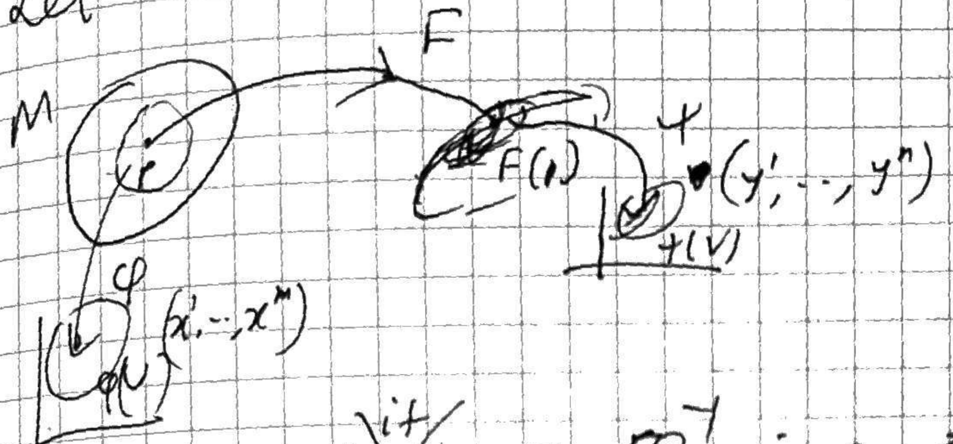
Manifolds and Smooth Mappings between them.

Let $F: M \to N$



m need not be the same as n

Def: F is smooth if $\psi \circ F \circ \varphi^{-1}$ is smooth in a neighborhood of $\varphi(p)$, for every p, and any chart φ, ψ including p and $F(p)$, respectively.

Def: If F and F^{-1} are both defined and are smooth, it is a diffeomorphism.

At any point $p \in M$, we can define a push forward of a vector v_p from T_pM to $T_{F(p)}N$.

Def: The linear map $F_*: T_pM \to T_{F(p)}N$ defined by $F_* v_p(f) = v_p(f \circ F)$, for any smooth $f: N \to \mathbb{R}$.

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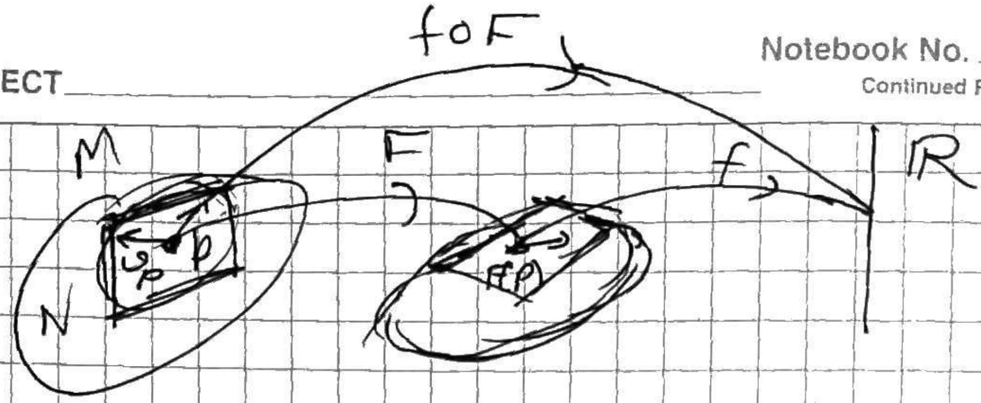
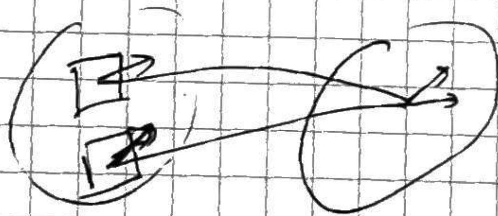


In coordinates, with slightly sloppy notation,

$F_* v_p(f) =$ [redacted] $\underbrace{v^i \frac{\partial y^j}{\partial x^i}} \frac{\partial f}{\partial y^j}$

$$\left(F_* v_p\right)^j = \frac{\partial y^j}{\partial x^i} v^i = F^j_{\ i} v^i$$

Note, by the way, [redacted] we cannot always have push-forward



If F is a many to one map, we do not have a well defined a field on N.

The push forward of vector fields make sense if F is a diffeomorphism.

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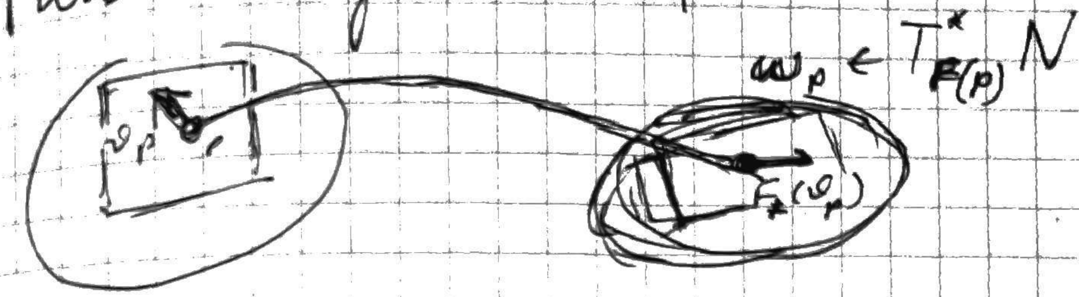
Pullback of a 1-form

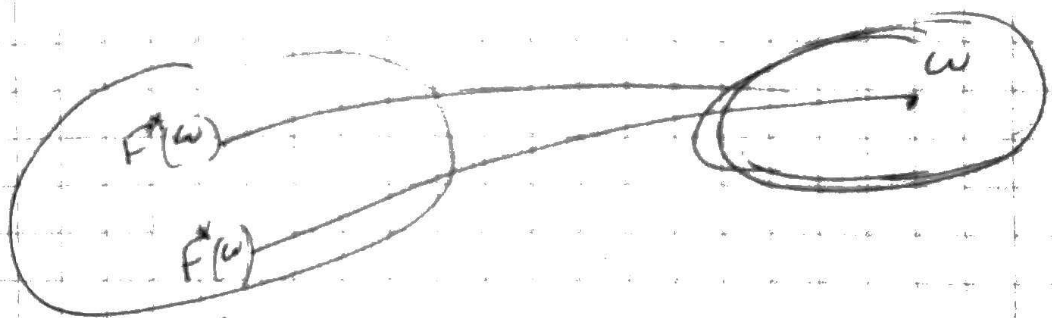


Def: The pullback $F^*\omega_p \in T^*_p M$ is defined by $(F^*\omega_p)(v_p)$ [redacted]

$$= \omega_p(F_*(v_p))$$

[redacted]

for any $\omega_p \in T^*_{F(p)} M$.



Notice that, for arbitrary F, we could define pullback of smooth 1-form fields from N to M. Contrast this with the requirement for push-forward of vector fields.

In coordinates

$$\omega_j dy^j \to (F^*\omega)_i dx^i = \omega_j \frac{\partial y^j}{\partial x^i} dx^i$$

$\omega_j(y^1(x), \ldots, y^n(x)) \dfrac{\partial y^j}{\partial x^i}(x)$ would be the pulled back smooth form on M.

Note that this is $\omega_j F^j_i$

In matrix language $(\underline{F_* v}) = \underline{F}\,\underline{v}$ ← column vector

$$(\underline{F^*\omega}) = \underline{\omega}\,\underline{F}$$

↑ rowvector

$\underline{F} \in \mathbb{R}^{n\times m}$

$\underline{v} \in \mathbb{R}^{m\times 1}$

$\underline{\omega} \in \mathbb{R}^{1\times n}$

By the way, any covariant tensor field could be pulled back: $T_{F(p)} \in T^{(s,0)}_{F(p)} N$

$$(F^* T_{F(p)})(v_1, \ldots, v_s) = T_{F(p)}(F_* v_1, \ldots, F_* v_s)$$

Using this local pullback, we can use this to define fields everywhere. In components

$$T_{j_1 \ldots j_s} \frac{\partial y^{j_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{j_s}}{\partial x^{i_s}}$$

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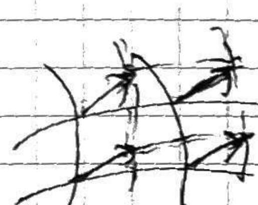
Now, let us go back to the map created by a vector field, namely $p \mapsto F_v(p,\lambda)$. The map $F_v(\cdot,\lambda)$ is a diffeomorphism from [redacted] M to M

We can define push forward of vectors $v_p \to F_* v$ and pull back of forms $\omega_p \to F^*\omega_p$. However if the local linear maps a full rank, we can also invert them.

Consider a small transformation



$$\underset{\text{Point}}{p} \xmapsto{\varphi} \underset{\text{coordinates}}{(x^1, \dots, x^n)}$$

$$F_v(p,\underset{\text{small }\lambda}{\varepsilon}) \xmapsto{\varphi} (x^1+\varepsilon v^1(x), \dots, x^n+\varepsilon v^n(x)) + O(\varepsilon^2)$$

Let us use the same coordinates: So $y^i = x^i + \varepsilon v^i + O(\varepsilon^2)$

$$F^j_i = \frac{\partial y^j}{\partial x^i} = \frac{\partial}{\partial x^i}\left\{x^j + \varepsilon v^j(x) + O(\varepsilon^2)\right\}$$

$$= \delta^j_i + \varepsilon \frac{\partial v^j}{\partial x^i} + O(\varepsilon^2)$$

$$e_i = \frac{\partial}{\partial x^i} \longmapsto F_* e_i = \left(\delta^j_i + \varepsilon \frac{\partial v^j}{\partial x^i}\right) e_j + O(\varepsilon^2)$$

[redacted]

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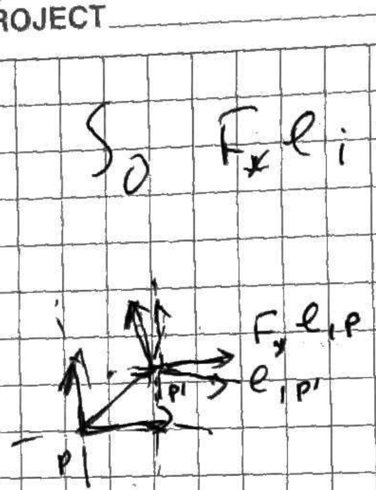
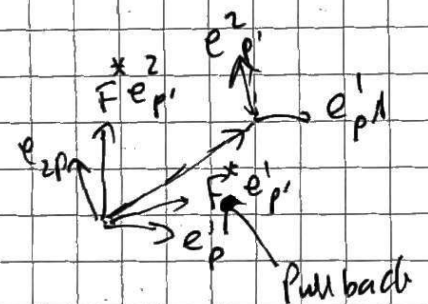
So $F_* e_i = e_i + \varepsilon \frac{\partial v^j}{\partial x^i} e_j + O(\varepsilon^2)$



For pullback of one forms! $F^* e^i$ [redacted] $= \left(\delta^i_j + \varepsilon \frac{\partial v^i}{\partial x^j}\right) e^j + O(\varepsilon^2)$



[redacted]f F^* is invertible

$F^{*-1} e^1_p$

[redacted] $F^{*-1} e^i = \left(\delta^i_j - \varepsilon \frac{\partial v^i}{\partial x^j}\right) e^j + O(\varepsilon^2)$

$= e^i - \varepsilon \frac{\partial v^i}{\partial x^j} e^j + O(\varepsilon^2)$

Note that $\langle F^{*-1} e^i, F_* e_j \rangle$ [redacted] $F(p,\varepsilon)$

$= \langle e^i - \varepsilon \frac{\partial v^i}{\partial x^k} e^k, e_j + \varepsilon \frac{\partial v^l}{\partial x^j} e_l \rangle + O(\varepsilon^2)$

$=$ [redacted] $\langle e^i, e_j \rangle + \langle e^i, \varepsilon \frac{\partial v^l}{\partial x^j} e_l \rangle + \langle -\varepsilon \frac{\partial v^i}{\partial x^k} e^k, e_j \rangle + O(\varepsilon^2)$

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$$= \delta^i_j + \varepsilon \frac{\partial v^l}{\partial x^j}\delta^i_l - \varepsilon \frac{\partial v^i}{\partial x^k}\delta^k_j + O(\varepsilon^2)$$

$$= \delta^i_j + \varepsilon \cancel{\frac{\partial v^i}{\partial x^j}} - \varepsilon \cancel{\frac{\partial v^i}{\partial x^j}} + O(\varepsilon^2)$$

Note the ~~to~~ leading order "push-forward" of the dual basis remains a dual basis.

So, using $F_p(\cdot, t)$, we can push forward a tensor from $T^{(s,r)}_p M$ to $T^{(s,r)}_{F(p)} M$.

One way is to [redacted] write the tensor as

$$T_{j_1 j_s}^{i_1 \cdots i_r}(x)\, e_{i_1}\otimes\cdots\otimes e_{i_r}\otimes e^{j_1}\otimes\cdots\otimes e^{j_s} \quad \text{at } p \mapsto (x^1,\dots,x^n).$$

and map it to $T^{i_1\cdots i_r}_{j_1 j_s}\, F_* e_{i_1}\otimes\cdots\otimes F_* e_{i_r}\otimes F_*^{-1} e^{j_1}\otimes\cdots\otimes F_*^{-1} e^{j_s}$

And compare it to $T^{i_1\cdots i_r}_{j_1 j_s}(x+\varepsilon v)\, e_{i_1}\otimes\cdots\otimes e_{i_r}\otimes e^{j_1}\otimes\cdots\otimes e^{j_s}$

In other words we want to calculate the derivative $\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[T_{F_v(p,\varepsilon)} - F^*_{(\cdot,\varepsilon)} T_p\right] = \mathcal{L}_v T$

This is called the Lie derivative ($\mathcal{L}_v$) w.r.t. the vector field v.

Let us get some idea about [redacted] what Lie derivative does with some lower level tensors. We will represent by [redacted] $e_i(\lambda)$ and [redacted] $e^i(\lambda)$, the vector and the form bases at $F_v(p, \lambda)$ $= p'$

1. $(0,0)$ order tensors or functions:

$$\mathcal{L}_v f = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[f(x+\varepsilon v) - f(x) \right]$$

$$= v^i \frac{\partial}{\partial x^i} f$$

It is just the directional derivative

2. Vector fields
If w is a vector field.

$$\mathcal{L}_v w = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \Big[w^i(x+\varepsilon v)\, e_i(\varepsilon) \quad \text{at } p' \to x+\varepsilon v$$

$$- w^i(x) \left(e_i(\varepsilon) + \varepsilon \frac{\partial v^j}{\partial x^i} e_j(\varepsilon) \right) \Big]$$

$$= v^j \frac{\partial w^i}{\partial x^j} e_i - w^i \frac{\partial v^j}{\partial x^i} e_j = \left(v^j \frac{\partial w^i}{\partial x^j} - w^j \frac{\partial v^i}{\partial x^j} \right) e_i$$

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Note that $\mathcal{L}_v w = -\mathcal{L}_w v$. Antisymmetry!

In fact $w(f) = w^i \frac{\partial}{\partial x^i} f$ is a function. If we apply $v(w(f)) = v^j \frac{\partial}{\partial x^j}\left(w^i \frac{\partial f}{\partial x^i}\right)$

$$= \left(v^j \frac{\partial}{\partial x^j} w^i\right) \frac{\partial}{\partial x^i} f + v^j w^i \frac{\partial^2 f}{\partial x^j \partial x^i}$$

Note that this is not a derivation because of the second derivative term and hence this operator on functions is not a vector field.

On the other [redacted] hand. $v(w(f)) - w(v(f))$

$$= \left(v^j \frac{\partial w^i}{\partial x^j} - w^j \frac{\partial v^i}{\partial x^j}\right) \frac{\partial f}{\partial x^i}$$

is a vector field's action on f.

So $\mathcal{L}_v w = [v, w]$

If $e_i = \frac{\partial}{\partial x^i}$ is a coordinate basis, what is $\mathcal{L}_{e_i} e_j$? Well $[e_i, e_j] = \left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0$

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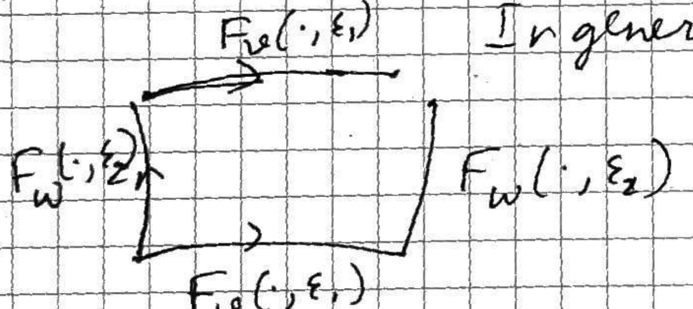
What is the map $F_{e_i}(\cdot, \lambda)$?

$$\frac{dx^i}{d\lambda} = 1 \qquad \frac{dx^j}{d\lambda} = 0 \text{ for } j \neq i$$

So $(x^1, \ldots, x^i, \ldots, x^n) \mapsto (x^1, \ldots, x^i + \lambda, \ldots, x^n)$

If we consider flows $F_{e_i}(\cdot, \varepsilon_1)$ and $F_{e_j}(\cdot, \varepsilon_2)$, they commute

$(x^1, \ldots, x^i, \ldots, x^j, \ldots, x^n)$ just goes [redacted] to $(x^1, \ldots, x^i + \varepsilon_1, \ldots, x^j + \varepsilon_2, \ldots, x^n)$.



In general it does not come to the same poin

Each coordinate x^i, thought of as a functio

$$x^i \to (F_v \circ F_w(\cdot, \varepsilon_2) - F_w \circ F_v)x_i = \varepsilon_1\varepsilon_2 [v, w]\, x^i$$

This is the mismatch of coordinates.

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If we need to check whether vector fields $v_1, \dots, v_n$ correspond to some $\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n}$ for some coordinate system $y^1, \dots, y^n$, the necessary condition is $[v_i, v_j] = 0$ for all pairs i, j.

[redacted]

3. Moving on to 1-forms

Basis at p

$$\mathcal{L}_v \omega = \lim \frac{1}{\varepsilon}\left[\omega_i(x+\varepsilon v)\, e^i(\varepsilon) - \omega_i(x)\left(e^i(\varepsilon) - \varepsilon \frac{\partial v^i}{\partial x^j} e^j(\varepsilon)\right)\right]$$

[redacted]

$$= \left(v^j \frac{\partial \omega_i}{\partial x^j} + \omega_j \frac{\partial v^j}{\partial x^i}\right) e^i$$

4. Mixed Tensor

$$\mathcal{L}_v T = \left(v^k \frac{\partial}{\partial x^k} T_{j_1 \dots j_s}^{i_1 \dots i_r} - T_{j_1 \dots j_s}^{k i_2 \dots i_r} \frac{\partial v^{i_1}}{\partial x^k} - T_{j_1 \dots j_s}^{i_1 k \dots i_r} \frac{\partial v^{i_2}}{\partial x^k} - \dots + T_{l j_2 \dots j_s}^{i_1 \dots i_r} \frac{\partial v^l}{\partial x^{j_1}} + \dots \right) e_{i_1} \otimes \dots \otimes e_{i_r} \otimes e^{j_1} \otimes \dots \otimes e^{j_s}$$

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