Differential 1-forms

The tangent space at point p at M, T_pM, is a linear vector space. We can construct a dual space T_p^*M with member

$$\omega : v \longmapsto \omega(v)$$

s.t. $\omega(a_1 v_1 + a_2 v_2) = a_1\omega(v_1) + a_2\omega(v_2)$

We would often use the notation $\langle \omega, v \rangle$

These objects are also called 1-form

Differential of a function:

Given a differentiable function f defined in some neighborhood ~~of~~ of the point p, we alway have a map) acting on [redacted] $v \in T_p(M)$ via $v_p \mapsto v_p(f)$. This is a linear functional.

We will let this map define a differential 1-form df, namely

$$df_p(v_p) = v_p(f)$$

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In the coordinate language

$$\sum_k v^k \frac{\partial}{\partial x^k} f$$

Note that $\langle dx^k, v \rangle$

$$= \sum_\ell v^\ell \frac{\partial}{\partial x^\ell} x^k$$

$$df = \sum_k \frac{\partial f}{\partial x^k} dx^k \qquad = \sum_\ell v^\ell \delta^k_\ell = v^k$$

In fact $\langle dx^k, \frac{\partial}{\partial x^\ell} \rangle = \delta^k_\ell$

For a general member $\omega \in T^*_p M$

$$\langle \omega, \frac{\partial}{\partial x^k} \rangle = \omega_k$$

and $\omega = \sum_k \omega_k \, dx^k$

$$\langle \omega, v \rangle = \langle \sum_k \omega_k \, dx^k, \sum_\ell v^\ell \frac{\partial}{\partial x^\ell} \rangle = \sum_k \omega_k v^\ell \delta^\ell_k$$

$$= \sum_k \omega_k v^k$$

How does ω components tranform

If we use a different coordinates $(x'^1, \ldots, x'^n)$,

$$\omega'_\ell = \langle \omega, \frac{\partial}{\partial x'^\ell} \rangle = \sum_k \omega_k \langle dx^k, \frac{\partial}{\partial x'^\ell} \rangle$$

$$= \sum_k \omega_k \frac{\partial x^k}{\partial x'^\ell}$$

Covariant vectors!

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Contrast with $v'^\ell = \sum_k v^k \frac{\partial x'^\ell}{\partial x^k}$

We can put together a space, like tangent bundle, a cotangent bundle T^*M.

We will use tangent bundle and cotangent bundles to define vector/covector fields as sections of the bundles, but before that, let us also set up higher tensors [redacted] and discuss fields all together. In particular we will discuss the symplectic form and metric tensor.

Let us specify the basis for T_pM and T_p^*M in a coordinate system $(x^1, \ldots, x^n)$. We will call $\frac{\partial}{\partial x^i}$ e_i, $v = \sum_i v^i e_i$, and dx^i e^i, $w = \sum_i w_i e^i$.

There are two ways of going about it. One is a basis dependent way. The other is via multilinear functions. For finite dimensional spaces these two approaches are equivalent a simple manner.

Let us start with a simple example of two finite dimensional vector spaces V and W with bases $\{e_i\}$, $\{f_j\}$

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$v \in V$ is written as $\sum_i v_i$[redacted]e_i and $w \in W$ is written as $\sum_j w_j f_j$ (I am not [redacted] bothering about upper or lower indices).

We define a vector space with basis vectors $e_i \otimes f_j$

$$v \otimes w = \sum_{ij} (v_i w_j)\, e_i \otimes f_j$$

Linear combination of such vectors [redacted] which are tensor products lead to vectors of the form $\sum_{ij} a_{ij}\, e_i \otimes f_j = a \in V \otimes W$

$V \otimes W$ is the tensor product of vector spaces V and W.

Note that a defines a multilinear map on [redacted] $V^* \times W^*$.

A multilinear map [redacted] $A : V_1 \times V_2 \to \mathbb{R}$ would have the property $A(a u_1 + b v_1, u_2) = a A(u_1, u_2) + b A(v_1, u_2)$ and $A(u_1, a u_2 + b v_2) = a A(u_1, u_2) + b A(u_1, v_2)$

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a defines a multilinear map thus

Λ_a [redacted] $(\lambda,\mu) = \sum_{ij} a_{ij}\,\lambda(e_i)\,\mu(f_j)$

One could have defined tensor products via this route as well.

In that case we just define $a : V \times W \to \mathbb{R}$

$(\lambda,\mu) \mapsto a(\lambda,\mu)$. If $\lambda = \sum_i$ [redacted] $_i\, e^i$ $\quad \mu = \sum_j \mu_j f^j$

When $\{e^i\}, \{f^j\}$ are the dual bases corresponding to $\{e_i\}, \{f_j\}$. We write $a = \sum_{ij} a^{ij}\, e_i \otimes f_j$

$$a(\lambda,\mu) = \sum_{ij} a^{ij}\,\lambda_i\,\mu_j$$

(Now we use subscripts and superscripts)

When we consider many vector spaces $V_1, \dots, V_k$, we can think of multilinear map from $V_1^* \times \cdots \times V_k^*$ to $\mathbb{R}$, defining $V_1 \otimes \cdots \otimes V_k$.

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Let us ~~come back~~ go to ~~manifolds~~ the a case where the vector spaces are just V & V^*. This is important for manifolds since we have T_pM and T_p^*M to build on.

Say V has a basis $(e_1, \dots, e_n)$ and V^* has ~~a~~ dual basis $(e^1, \dots, e^n)$ with

$$\langle e^j, e_k \rangle \equiv e^j(e_k) = \delta^j_k .$$

We define mixed tensors in the space

$$\underbrace{V \otimes \cdots \otimes V}_{p \text{ times}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{s \text{ times}}$$

with basis elements

$$e_{i_1 \dots i_p}{}^{j_1 \dots j_s} \equiv e_{i_1} \otimes \cdots \otimes e_{i_p} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s}$$

A tensor $T \equiv \sum_{\{i\}\{j\}} T^{i_1 \dots i_p}{}_{j_1 \dots j_s} \, e_{i_1 \dots i_p}{}^{j_1 \dots j_s}$

This $\binom{p}{s}$ tensor is also a map from

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$$T: \overbrace{V^* \times \cdots V^*}^{p \text{ times}} \times \overbrace{V \times \cdots \times V}^{s \text{ times}} \to \mathbb{R}$$

$$T(\omega^{(1)}, \ldots, \omega^{(p)}, v_{(1)}, \ldots, v_{(s)})$$

$$= \sum_{\{i\}\{j\}} T^{i_1 \cdots i_p}_{j_1 \cdots j_s} \, \omega^{(1)}_{i_1} \cdots \omega^{(p)}_{i_p} \, v^{j_1}_{(1)} \cdots v^{j_s}_{(s)}$$

Now you know why physicists love summation convention: repeated indices are summed! By the way, my convention is slightly different from the book's. The book uses $T(v_{(1)}, \ldots, v_{(s)} \mid \omega^{(1)} \ldots \omega^{(p)})$ instead.

We call the vector space of $\binom{p}{s}$ tensors at point p $T^{p,s}_p M$

Note that an $\binom{p}{s}$ tensor and a vector can define an $\binom{p}{s-1}$ tensor $T(\cdots, \ldots, v, \ldots)$

Or in indices $\sum_{j_k} T^{i_1 \cdots i_p}_{j_1 \cdots j_k \cdots j_s} v^{j_k} = S^{i_1 \cdots i_p}_{j_1 \cdots j_{k-1} j_{k+1} \cdots j_s}$

I have a choice which inde

Same way a form ω and T leads to an $\binom{p-1}{s}$ tensor $T(\ldots, \omega, \ldots, \ldots)$

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One can also contract an upper index with a lower index giving rise to a $\binom{p-1}{q-1}$ tensor

$$\sum_{j_k i_l} T^{i_1 \cdots i_l \cdots i_p}_{j_1 \cdots j_k \cdots j_q} \, \delta^{j_k}_{i_l}$$

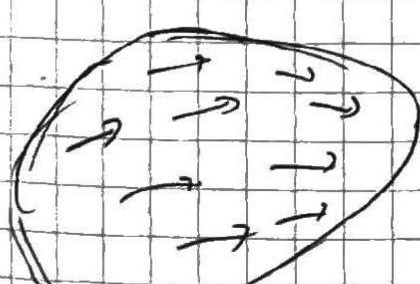
One can think of this as

$$\sum_i T(\ldots, e^i, \ldots, \ldots, e_i, \ldots)$$

To go over to manifolds, we just use $e_i = \frac{\partial}{\partial x^i}$, $e^i = dx^i$.

Vector and tensor fields

Def: ~~A~~ A vector field v is a map from $M \to TM$ satisfying $p \mapsto (p, v_p)$ where $v_p \in T_pM$



These things should be thought of as sections in a vector bundle.

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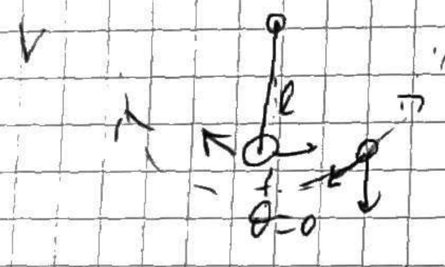
Before going on, let us consider a very simple example: a pendulum ~~with~~ affected by strong drag

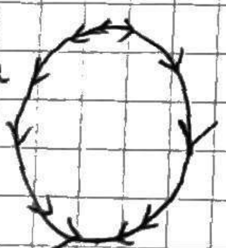


$$ml^2\ddot{\theta} = -l\Gamma\dot{\theta} - mgl\sin\theta$$

If Γ is very large, we ignore the inertial term

$$\dot{\theta} = -\frac{mg}{\Gamma}\sin\theta = -A\sin\theta$$

Configuration Space

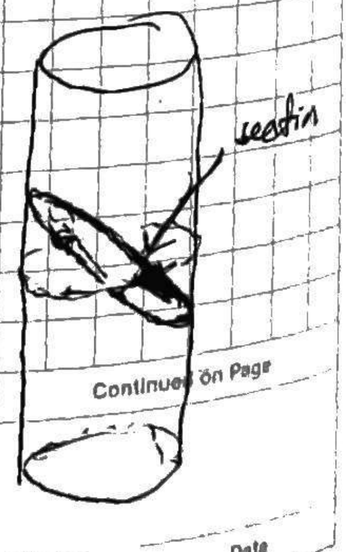


This is a vector field on S^1

~~[illegible]~~ What does the general position, velocity space look like?

$S^1 \times \mathbb{R}$

The flow



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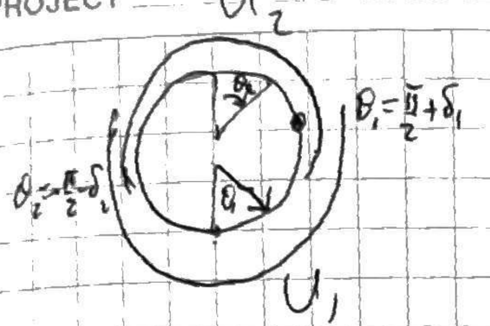
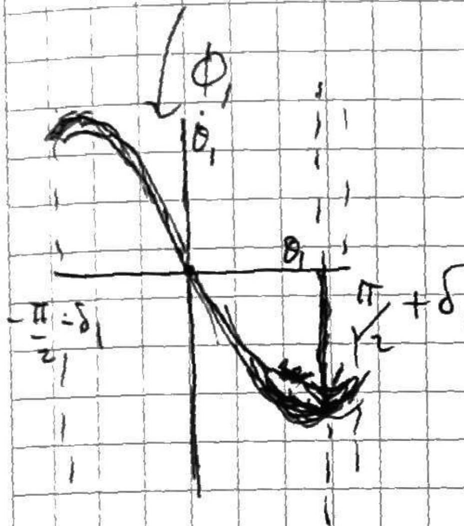
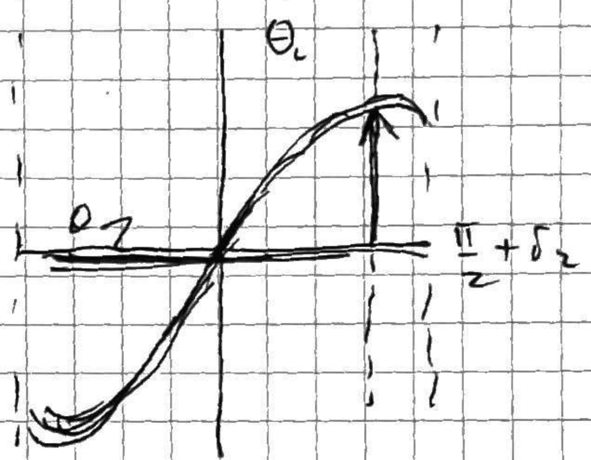
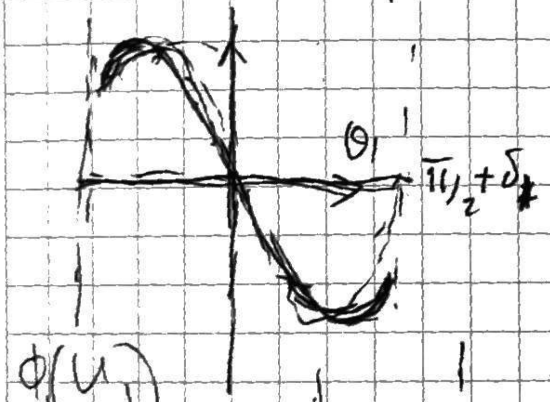


How does it look in a coordinate Sazstem



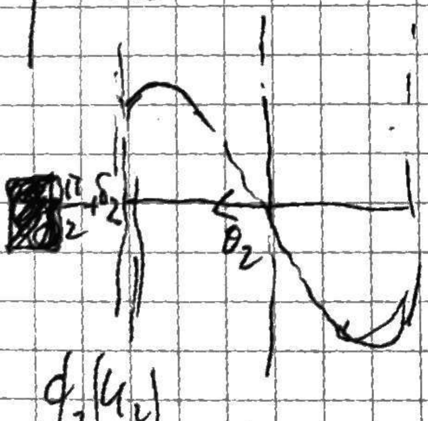


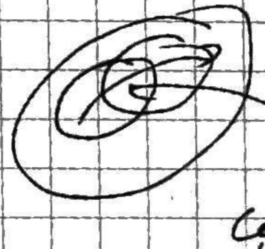
In that overlap $\theta_2 = \pi - \theta_1 \Rightarrow \dot\theta_2 = -\dot\theta_1$



$\theta_2 = \pi - \theta_1$, $v_2 = -v_1$

flip





Need to change coordinates and vector components

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In this case (S^1), we could chose better coordinates but not in the case of sphere S^2.

Locally, the vector field is described as

$(x^1, \dots, x^n, v^1(x^1, \dots, x^n), \dots, v^n(x^1, \dots, x^n))$

For a C^∞ vector field on a C^∞ manifold, the functions $v^i(x^1, \dots, x^n)$ would have to be C^∞ functions of the coordinates. One can show that in other charts, they have to be C^∞ functions as well: namely, $v'^i(x'^1, \dots, x'^n)$, are C^∞ functions of $x'^1, \dots, x'^n$.

$$v'^i = \sum_j v^j(x^1(x'^1, \dots, x'^n), \dots, x^n(x'^1, \dots, x'^n)) \frac{\partial x^i}{\partial x'^j}$$

$x \to x'$ and $x' \to x$ maps being C^∞ and invertible guarantees.

By the way, the tensor $T = \sum_{\{i\}\{j\}} T^{i_1 \dots i_r}_{j_1 \dots j_s} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s}$

$$= \sum_{\{k\}\{l\}} \left(\sum_{\{i\}\{j\}} T^{i_1 \dots i_r}_{j_1 \dots j_s} \frac{\partial x'^{k_1}}{\partial x^{i_1}} \cdots \frac{\partial x'^{k_r}}{\partial x^{i_r}} \frac{\partial x^{j_1}}{\partial x'^{l_1}} \cdots \frac{\partial x^{j_s}}{\partial x'^{l_s}} \right) \frac{\partial}{\partial x'^{k_1}} \otimes \dots \otimes \frac{\partial}{\partial x'^{k_r}} \otimes dx'^{l_1} \otimes \dots \otimes dx'^{l_s}$$

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$$= \sum_{\{k\}\{l\}} T'_{l_1 \cdots l_s}{}^{k_1 \cdots k_r} \frac{\partial}{\partial x'^{k_1}} \otimes \cdots \otimes \frac{\partial}{\partial x'^{k_r}} \otimes dx'^{l_1} \otimes \cdots \otimes dx'^{l_s}$$

This equation tells you how the n^{s+r} T components transform.

Like TM, we can define $T^{(s,r)}M$ attaching to each point $p \in M$, $T_p^{(s,r)}M$. Local coordinates for $T^{(s,r)}M$ given by $(x^1, \dots, x^n, \{t_{\{l\}}{}^{\{k\}}\})$ with tranformation rules specified between $(U, \mathbb{R}^{n^{s+r}})$ and $(V, \mathbb{R}^{n^{s+r}})$

We can also define C^∞ tensor fields by defining $T: M \to T_p^{sr}M$, with the components being C^∞ in each chart.

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Before we move on, useful formulea for transformation

$$e^i = dx^i = \frac{\partial x^i}{\partial x'^j} dx'^j = \frac{\partial x^i}{\partial x'^j} e'^j$$

$$e'^i = dx'^i = \frac{\partial x'^i}{\partial x^j} dx^j = \frac{\partial x'^i}{\partial x^j} e^j$$

$$e_i = \frac{\partial}{\partial x^i} = \frac{\partial x'^j}{\partial x^i} \frac{\partial}{\partial x'^j} = \frac{\partial x'^j}{\partial x^i} e'_j$$

$$e'_i = \frac{\partial}{\partial x'^i} = \frac{\partial x^j}{\partial x'^i} \frac{\partial}{\partial x^j} = \frac{\partial x^j}{\partial x'^i} e_j$$

We are using summation convention from now on. Repeated indices are expected to be summed.

So $v^i e_i = v^i \frac{\partial x'^j}{\partial x^i} e'_j = v'^j e'_j \Rightarrow v'^j = v^i \frac{\partial x'^j}{\partial x^i}$

$\omega_i e^i = \omega_i \frac{\partial x^i}{\partial x'^j} e'^j = \omega'_j e'^j \Rightarrow \omega'_j = \omega_i \frac{\partial x^i}{\partial x'^j}$

etc. etc.

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