Preliminaries

Sets

$x \in A$

element ↑   ↑ set

Venn diagrams

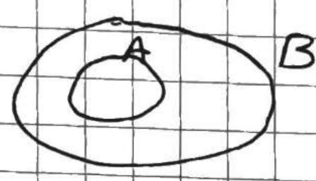
Sometimes one represents sets as

$A = \{x \mid x \text{ satisfies property } P\}$

Notation review:

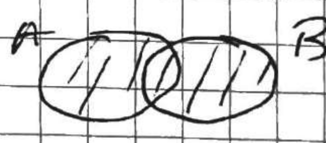
$A = B$ the sets have exactly the same elements

$A \subset B$



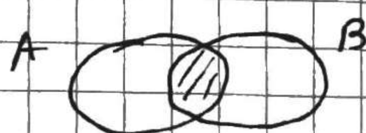
$A \supset B$ same as $B \subset A$

$A \cup B$



union

$A \cap B$



intersection

$A - B$

difference

$A \times B$ $\{(x,y) \mid x \in A, y \in B\}$

set of ordered pairs

Cartesian product

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Notebook No.

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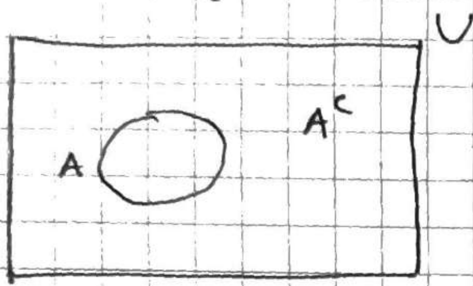
The set with no element ϕ null set

Sometimes, the [redacted] topic has a natural 'universal set' U, so that all the sets under discussions are subsets of U.



$A^c = U - A$

Complement of A.

Number system:
(Vaughn's notation)

$\mathbb{Z}_* = \{1, 2, 3, \cdots\} = \mathbb{N}$ Positive integers
Natural numbers

$\mathbb{Z}_+ = \{0, 1, 2, 3, \cdots\} = \mathbb{N}_0$

$\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$ Integers

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$\mathbb{Q} = \{p/q \mid p \in \mathbb{Z},\ q \in \mathbb{Z},\ q \neq 0\}$ Rational Numbers

$\mathbb{R}$, Real numbers $\qquad \mathbb{Q} \subset \mathbb{R}$

One can start with Peano axioms for natural [redacted] numbers and construct all the other sets like $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ etc. In fact, one can start with null set ϕ and construct natural numbers. For our purpose, we start with rules for $\mathbb{Q}$ and $\mathbb{R}$ and regard $\mathbb{N}$, $\mathbb{Z}$ etc to be special subsets of $\mathbb{Q}$.

Both $\mathbb{Q}$ and $\mathbb{R}$ are ordered fields.

Field [redacted] definition [Physicists use field to mean something else, e.g. electric field]

A field [redacted] is a set F with operations $+$ and $\cdot$, addition and multiplication, respectively, with properties

1) Associativity: $a+(b+c) = (a+b)+c$, $a\cdot(b\cdot c) = (a\cdot b)\cdot c$

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2) commutativity $a+b=b+a$, $a \cdot b = b \cdot a$

3) Additive & multiplicative identities,
$0, 1 \in F$, $0 \neq 1$, $a+0=a$, $a \cdot 1 = a$

4) Additive inverse: $\forall a \in F$, $\exists (-a)$, such that
$a+(-a)=0$

5) [redacted] Multiplicative inverse: $\forall a \in F$, $a \neq 0$,
$\exists a^{-1}$, such that, $a \cdot a^{-1} = 1$

6) Distributive law: $a \cdot (b+c) = ab + a \cdot c$

$\mathbb{Q}$ & $\mathbb{R}$ are ordered fields

[redacted] Total order on [redacted] a set A

$\forall x, y \in A$ $x \leq y$ and $y \leq x \Rightarrow x = y$ (Antisymmetry)

$\forall x, y, z \in A$ $x \leq y$, $y \leq z \Rightarrow x \leq z$ (Transitivity)

$\forall x, y \in A$ $x \leq y$ or $y \leq x$ (Connexity)

$x \leq y$ is the same as $y \geq x$

Ordered field F

$a, b, c \in F$ 1) If $a \geq b$ then $a+c \geq b+c$

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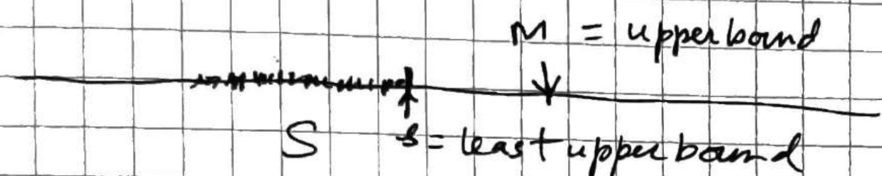
2) If $a \geq 0$, $b \geq 0$ then $ab \geq 0$

The big difference between $\mathbb{Q}$ and $\mathbb{R}$ is that $\mathbb{R}$ is complete, it has no holes.

Dedekind-completeness of $\mathbb{R}$: Any non-empty subset of $\mathbb{R}$ that is bounded above, has a least upper bound in $\mathbb{R}$.



Our textbook uses convergence of Cauchy sequences in $\mathbb{R}$, which is equivalent to Dedekind-completeness.
The least upperbound is also called supremum; $s = \sup S$

$\mathbb{Q}$ does not have this property.

Consider $S_1 = \{ r \in \mathbb{Q} \mid r \geq 0, r^2 < 1 \}$

$S_2 = \{ r \in \mathbb{Q} \mid r \geq 0, r^2 < 2 \}$

What are $\sup S_1$ and $\sup S_2$?

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We will use $|x| = \max\{x, -x\}$ for $x \in \mathbb{R}$

$|z| = \sqrt{x^2 + y^2}$ for $z = x+iy \in \mathbb{C}$

Def: The sequence $\{z_n\}$ is bounded if there exist a positive real M s.t. $|z_n| < M$ for all n

Def: A [redacted] real sequence $\{x_n\}$ is [redacted] increasing ([redacted] decreasing) is $x_{n+1} \geq x_n$ ($x_{n+1} \leq x_n$).

If it is strictly increasing (decreasing) [redacted] ... $x_{n+1} > x_n$ ($x_{n+1} < x_n$)

Increasing or decreasing sequences are called monotonic.

Def: A sequence $\{z_n\}$ is cauchy iff, for any $\varepsilon > 0$, there is positive integer N s.t. for any $p,q > N$, $|z_p - z_q| < \varepsilon$.

Def: The sequence $\{z_n\}$ converges to the limit z [redacted] (written $\{z_n\} \to z$) if for every $\varepsilon > 0$, there is [redacted] a positive integer N, s.t., for an $n > N$, $|z_n - z| < \varepsilon$.

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