Course: Mathematical Physics

Problem Set 2

Due: Oct 23, 2019

1. Find the radius of convergence of the following power series
a) $\sum_{k=0}^{\infty} \frac{z^{2k}}{(k!)^2}$, b) $\sum_{k=0}^{\infty} z^{2k}$, c) $\sum_{k=0}^{\infty} (k!)^2 z^{2k}$.

2. With $\theta \in [-\pi, \pi]$, $g_n(\theta) = \frac{1}{2\pi} \frac{(z\cos\theta)^n}{n!}$, $n = 0, 1, 2, 3..$
and $f(\theta) = \frac{1}{2\pi} e^{z\cos\theta}$ for some $z \in \mathbb{C}$.
Show that $\sum_n g_n$ converges uniformly to f in $[\pi, \pi]$

3. If a series of integrable functions g_n, $\sum_n g_n$, converges uniformly to a function f in $[a, b]$ then f turns out to be integrable and $\sum_n \int_a^b g_k(x)\,dx$ converges to $\int_a^b f(x)\,dx$. Using this result, show that $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{z\cos\theta}\,d\theta = \sum_{k=0}^{\infty} \frac{z^{2k}}{(k!)^2}$.

4. Consider the integral $I(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x\cos\theta}\,d\theta = \frac{e^x}{2\pi}\int_{-\pi}^{\pi} e^{x(\cos\theta - 1)}\,d\theta$
for large positive real x.
a) Plot $e^{x(\cos\theta - 1)}$ as a function of $\theta \in [-\pi, \pi]$, for $x = 1$ and $x = 10$.
b) Perform an asymptotic expansion $I(x)$ for large positive x: $I(x) \sim \frac{e^x}{\sqrt{x}}\left[a + \frac{b}{x} + O\left(\frac{1}{x^2}\right)\right]$. Determine a and b.