Midterm: Mathematical physics 30th Oct 2019

Due: Nov 1st, 2019 1:40PM

1. Consider $f : \mathbb{C} \times \mathbb{H} \to \mathbb{C}$ defined by

$$f(z,\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau + 2\pi i n z}$$
$$= 1 + \sum_{n=1}^{\infty} e^{\pi i n^2 \tau}\left(e^{+2\pi i n z} + e^{-2\pi i n z}\right)$$

$\mathbb{H} =$ [redacted] $\{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$

a) Show that series $1 + \sum_{n=1}^{\infty} e^{\pi i n^2 \tau}\left(e^{2\pi i n z} + e^{-2\pi i n z}\right)$ is absolutely convergent for any $\tau \in \mathbb{H}$ and any $z \in \mathbb{C}$.

[We organize it as $1 + e^{\pi i \tau} e^{2\pi i z} + e^{\pi i \tau} e^{-2\pi i z} + e^{4\pi i \tau} e^{4\pi i z} + e^{4\pi i \tau} e^{-4\pi i z} + \cdots$] (5)

b) Show that $f(z+1, \tau) = f(z,\tau)$. (1)

c) Show that $f(z+\tau, \tau) = e^{-\pi i \tau - 2\pi i z} f(z,\tau)$ (4)

2. Consider finite dimensional unitary vector spaces (over $\mathbb{C}$) V and W, $\dim V = m$ and $\dim W = n$.

$A : V \to W$ is a linear map.

We define $A^{+} : W \to V$ via

$$\langle y, Ax \rangle_W = \langle A^{+} y, x \rangle_V$$

where $\langle\, , \rangle_V$ and $\langle\, , \rangle_W$ are the scalar products in the respective spaces.

a) If $n > m$ and $\operatorname{rank}(A) = m$, what is the nullity of A^{+} (namely, the dimension of the kernel of A^{+})? 3

b) $V = \mathbb{C}^2$, $W = \mathbb{C}^3$. The matrix for A in the standard basis system, with the standard scalar product, being given by

$$\underline{A} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 0 & 2 \end{bmatrix}.$$

The matrix of A^{+} is given by $\underline{A}^{+} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$.

Find the eigenvalues of [redacted] $A^{+}A$ and of AA^{+}. 4

c) Show that, if $n > m$, and if eigenvalues of $A^{+}A$ are $\lambda_1, \ldots, \lambda_m$, each nonnegative, then the eigenvalues of AA^{+} are [redacted] $\lambda_1, \ldots, \lambda_m$, and $n-m$ zeros. 3

d) Consider $\underline{A} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 0 & 2 \end{pmatrix}$

Note that $\underline{A} = \sqrt{6}\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}\left(\frac{1}{\sqrt{2}} \;\; \frac{1}{\sqrt{2}}\right) + 2\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}\left(\frac{1}{\sqrt{2}} \;\; -\frac{1}{\sqrt{2}}\right)$

Compute $\underline{A}^{+}A$ and show that it is

$$6\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}\left(\frac{1}{\sqrt{2}} \;\; \frac{1}{\sqrt{2}}\right) + 4\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}\left(\frac{1}{\sqrt{2}} \;\; -\frac{1}{2}\right)$$

Compute AA^{+} and show that it is

$$6\begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}\left(\frac{1}{\sqrt{3}} \;\; \frac{1}{\sqrt{3}} \;\; \frac{1}{\sqrt{3}}\right) + 4\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}\left(\frac{1}{\sqrt{2}} \;\; 0 \;\; -\frac{1}{\sqrt{2}}\right)$$

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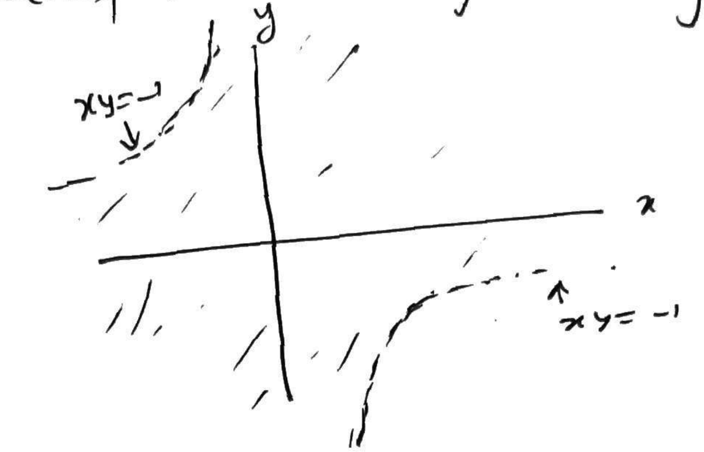
3. We define a manifold in 2d by the region

$1 + xy > 0$



We have a (second order symmetric) covariant tensor with components

$$g_{xx} = 0, \quad g_{yy} = 0, \quad g_{xy} = g_{yx} = \frac{1}{1+xy}$$

$$g = \frac{1}{2} g_{\mu\nu}\, dx^\mu \otimes dx^\nu = \frac{dx \otimes dy}{1+xy} \qquad (\mu, \nu = 1, 2)$$

a) Define a chart in the region $x > 0,\ y > 0$

Via $x = e^{r+t}, \quad y = e^{r-t}$

What does the tensor g components in the r, t system look like. (5)

b) Consider the vector field $v = \frac{1}{2}\left(x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}\right)$

Express this vector field in r, t system expanded in the basis $\frac{\partial}{\partial r}, \frac{\partial}{\partial t}$. (5)

c) Calculate $\mathcal{L}_v g$ explicity in the x, y system

The Lie derivative of this tensor has components

$$\left(\mathcal{L}_v g\right)_{\mu\nu} = v^\rho \frac{\partial g_{\mu\nu}}{\partial x^\rho} + g_{\mu\rho} \frac{\partial v^\rho}{\partial x^\nu} + g_{\rho\nu} \frac{\partial v^\rho}{\partial x^\mu}$$

What would the computation be in r, t coordinates? (5)

d) Find integral curves for the vector field v.

Describe them in the x, y coordinate system. (5)

e) Consider the two-form

$$\Omega = \frac{dx \wedge dy}{1 + xy} \qquad \Omega_{xy} = -\Omega_{yx}$$

$$\Omega_{xx} = \Omega_{yy} = 0$$

Compute $\mathcal{L}_v \Omega$ (rules same as g's). (5)