Mathematical Physics 464/511
Final Exam, Dec 2019 [Total 100]

1. Consider the analytic function $f$ defined on $\mathcal{R} = \{z \in \mathbb{C} \mid \cosh z \neq 0\}$ via
$$f(z) = \operatorname{sech} z = \frac{1}{\cosh z} = \frac{2}{e^{z} + e^{-z}}$$
for $z \in \mathcal{R}$.

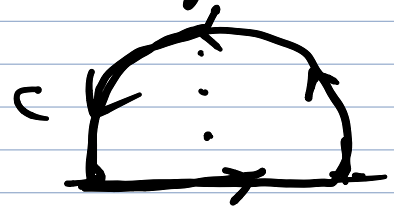
a) Find the singularities of $f$ in $\mathbb{C}$ and comment on them. [10]

b) Consider $g$ defined by
$$g(z) = \pi \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)}{z^2 + (n+\frac{1}{2})^2 \pi^2}$$
at any $z \neq \pm\frac{\pi i}{2}, \pm\frac{3\pi i}{2}, \pm\frac{5\pi i}{2}, \ldots$

Show that $f$ and $g$ have the same singularities. [10]

c) Let us evaluate
$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} \operatorname{sech} t \, dt \qquad \omega \in \mathbb{R}$$

by computing $I(\omega) = \oint_C e^{i\omega z}\,\mathrm{sech}\, z \, dz$




for $\omega > 0$. Find contributions to $I(\omega)$ from residues in the upper half plane.
Sum the series. [10]

2. We have seen the geodesic equation derived from minimizing

$$S_0[x] = \int_1^2 \sqrt{g_{ij}(x)\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}}\, d\lambda$$

[summation convention in use!]

but we needed to deal with reparametrization invariance by fixing $g_{ij}(x)\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda} = \text{constant}$.

Alternatively, we could derive the geodesic equation from minimizing

$$S_1[x] = \frac{1}{2}\int_1^2 g_{ij}(x)\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}\, d\lambda$$

giving rise to

$$\frac{d}{d\lambda}\left(g_{ij}\frac{dx^j}{d\lambda}\right) - \frac{1}{2}\frac{\partial g_{kl}}{\partial x^i}\frac{dx^k}{d\lambda}\frac{dx^l}{d\lambda} = 0$$

as the Euler-Lagrange equation.

a) Consider the half space

$$H = \{(x^1, x^2) \in \mathbb{R}^2 \mid x^2 > 0\}$$

The metric tensor is

$$\underline{g}(x) = \begin{pmatrix} g_{11}(x) & g_{12}(x) \\ g_{21}(x) & g_{22}(x) \end{pmatrix} = \begin{pmatrix} \frac{1}{(x^2)^2} & 0 \\ 0 & \frac{1}{(x^2)^2} \end{pmatrix}$$

Write down the geodesic equations.

[10]

b) $$S_1[x] = \frac{1}{2}\int_1^2 \frac{1}{(x^2)^2}\left[\left(\frac{dx^1}{d\lambda}\right)^2 + \left(\frac{dx^2}{d\lambda}\right)^2\right] d\lambda$$

has symmetries $x^1 \to x^1 + a$ and $\lambda \to \lambda + c$, giving us conservation laws

$$\frac{d}{d\lambda}\left(\frac{1}{(x^2)^2}\frac{dx^1}{d\lambda}\right) = 0$$

and
$$\frac{d}{d\lambda}\left[\frac{1}{(x^2)^2}\left\{\left(\frac{dx^1}{d\lambda}\right)^2 + \left(\frac{dx^2}{d\lambda}\right)^2\right\}\right] = 0$$

Set $\frac{1}{(x^2)^2}\frac{dx^1}{d\lambda} = \frac{1}{R} = \text{const.}$

$$\frac{1}{(x^2)^2}\left[\left(\frac{dx^1}{d\lambda}\right)^2 + \left(\frac{dx^2}{d\lambda}\right)^2\right] = 1 \quad \text{(by scaling } \lambda\text{)}$$

Derive a differential equation for $\frac{dx^2}{dx^1}$ in terms of $x^1, x^2$ (but not $\lambda$).

[10]

c) Solve the last equation to obtain a relation between $x^1, x^2$ for general geodesics.

[10]

3. a) Solve the differential equation

$$\frac{d}{dt}\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}, \quad t \in [0, \infty)$$

with initial condition $\begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

[10]

b) What is the maximum value of $u_1(t)$ for $t \in [0, \infty]$?

[5]

c) As $t \to \infty$, what happens to $\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$?

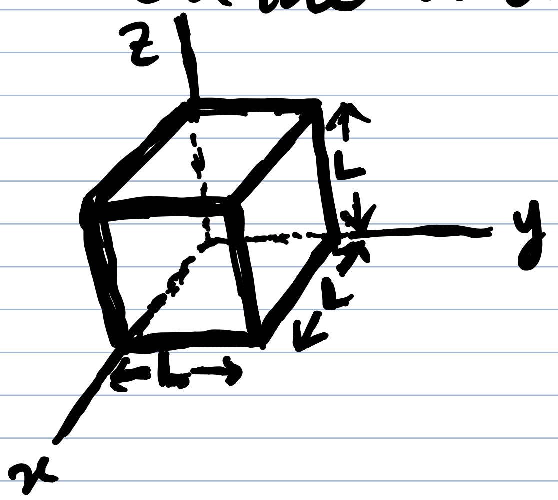
[5]

4. The diffusion equation for the time-dependent density of particles $\rho(x, y, z, t)$ is given by

$$\frac{\partial \rho}{\partial t} = D\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\rho.$$

The particles are inside an $L\times L\times L$ box, with
$0\le x\le L$, $0\le y\le L$, $0\le z\le L$.

The boundary condition is absorbing, meaning $\rho=0$ on the walls of the box.




With such boundary conditions, the total number of particles can change.

a) We start with $N_0$ particles at $t=0$.

$$\rho(x,y,z,0)=A\sin\frac{\pi x}{L}\sin\frac{\pi y}{L}\sin\frac{\pi z}{L}$$

Relate $A$ to $N_0$, using $N_0=\int\rho(x,y,z,0)\,dx\,dy\,dz$.
Solve for $\rho(x,y,z,t)$. Remember that $D$ is a constant (the diffusion constant).

[10]

b) The time-dependent number of particles in the box

$$N(t) = \int_0^L dx \int_0^L dy \int_0^L dz \; \rho(x, y, z, t)$$

keeps decreasing with time $t$.

At what $t$ does $N(t)$ become $N_0/2$?

[10]