Quantum Mechanics and Atomic Physics

Lecture 14:
Review for Midterm Exam

http://www.physics.rutgers.edu/ugrad/361

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Midterm Exam Info

- **Midterm exam Wed. Oct. 23**
  - During regular class time (1:40-3pm) and room
  - chapters 1-4, homeworks 1-5 and related lectures
  - it will be closed book but you are allowed to bring one equation sheet
    - one 8.5" x 11" sheet of paper with formulas and notes to consult during the exam. You may write on both sides of this cheat sheet.
    - you should also bring a couple of pencils and a scientific calculator.
  - check website for more information.

- Todays class will be a midterm exam review session.
- Slides posted on course webpage.
How to study

- Read chapters 1-4
  - And the examples in the book!
- Study related lectures
- Review all homework problems in homeworks 1-5
- Go over review session material
- Prepare your formula sheet
- Form study groups
Make sure you know how to draw wave functions and probability distributions for known potentials we solved for in class!

(note: Harmonic Oscillator not on exam)
Recall useful animations!

- https://phet.colorado.edu/en/simulation/legacy/quantum-tunneling
- http://phet.colorado.edu/simulations/sims.php?sim=Quantum_Bound_States
Example 1: Bohr Theory

A particle of mass $m$ moves in a circular orbit in a potential $V = V_0 \frac{r}{a}$, where $V_0$ and $a$ are constants, and $r$ is the orbit radius, so the force is $|F(r)| = \frac{V_0}{a}$. Using the Bohr model, find the energy levels in terms of $V_0$, $a$, $m$, $\hbar$, and a quantum number $n$. The final answer must not have $v$ or $r$ in it.
\[ F(r) = \frac{V_0}{a} = \frac{mv^2}{r} \]
\[ \Rightarrow r = \frac{amv^2}{V_0} \]

Bohr model: \[ L = mv^2r = n\hbar \]
\[ \Rightarrow v = \frac{\hbar}{mr} \]
\[ \Rightarrow v = \frac{n\hbar}{m} \frac{V_0}{amv^2} \]
\[ \Rightarrow v^2 = \frac{n\hbar V_0}{am^2} \Rightarrow v = \left( \frac{n\hbar V_0}{am^2} \right)^{\frac{1}{2}} \]

And, \[ v = V_0 \frac{E}{a} = \frac{V_0}{a} \frac{amv^2}{V_0} = mv^2 \]
\[ \Rightarrow v = mv^2 \]

So, \[ E = KE + PE = \frac{1}{2}mv^2 + mv^2 = \frac{3}{2}mv^2 \]
\[ E = \frac{3}{2}m \left( \frac{n\hbar V_0}{am^2} \right)^{\frac{2}{3}} \]
\[ \Rightarrow E = \frac{3}{2} \left( \frac{n\hbar V_0}{am^2} \right)^{\frac{2}{3}} \]
General case:

\[ V(r) = V_0 \left( \frac{r}{a} \right)^k \]

\[ F(r) = -\frac{dV}{dr} = -k \cdot V_0 \left( \frac{r}{a} \right)^{k-1} \left( \frac{1}{a} \right) \]

And a similar analysis gives:

\[ E = \left( 1 + \frac{k}{2} \right) \left[ \frac{n^2 \frac{\hbar^2}{2m} \left( \frac{V_0}{a} \right)^k}{k \cdot m a^2} \right]^{\frac{1}{k+2}} \]

So, for \( k = 1 \):

\[ E = \frac{3}{2} \left( \frac{n^2 \frac{\hbar^2}{2m} \frac{V_0^2}{a^2}}{m a^2} \right)^{1/3} \]
Example 2: Normalized Wave Function and Probability

- A particle is described by the wavefunction:

\[
\Psi(x) = A \frac{1}{x + 2i}, \text{ for } -\infty \leq x \leq \infty
\]

- (a) Determine the constant A.

- (b) What is the probability that the particle will be found between \( x = 0 \) and \( x = 1 \)?
Let’s use:

\[
\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)
\]

(a) \( \psi(x) = A \frac{1}{x+2i}, \quad \int_{-\infty}^{\infty} \psi^*(x) \psi(x) = 1 \)

\[
A^2 \int_{-\infty}^{\infty} \frac{1}{(x-2i)(x+2i)} dx = 1
\]

\[
A^2 \int_{-\infty}^{\infty} \frac{1}{x^2 + 4} dx = A^2 \left[ \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) \right]_{-\infty}^{\infty} = 1
\]

\[
A^2 \left[ \tan^{-1}(\infty) - \tan^{-1}(-\infty) \right] = 1
\]

\[
\frac{A^2}{2} \left( \frac{\pi}{2} - \left(\frac{-\pi}{2}\right) \right) = \frac{A^2}{2} \pi = 1 \quad \Rightarrow \quad A = \sqrt{\frac{2}{\pi}}
\]

(b) \( \int_{0}^{1} \psi^*(x) \psi(x) = \ldots = \frac{2}{\pi} \left[ \frac{1}{2} \tan^{-1} \left( \frac{x}{2} \right) \right]_0^1
\]

\[
= \frac{1}{\pi} \left[ \tan^{-1}\left(\frac{1}{2}\right) - \tan^{-1}(0) \right] = \frac{1}{\pi} (0.464 - 0)
\]

\[
= 0.1476 \Rightarrow 14.8^\circ
\]
Example 3: Infinite Square Well and expectation values

Find the expectation value of the momentum and energy for the hypothetical eigenfunction above. Use the results to determine the uncertainty in the momentum.

\[
\psi(x) = \sqrt{\frac{2}{3L}} \left(1 - \cos \frac{4\pi x}{L}\right)
\]

\[
E = \frac{p^2}{2m} \quad p_x = -i\hbar \frac{d}{dx}
\]

\[
E_{op} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2}
\]

\[
\frac{d\psi}{dx} = \sqrt{\frac{2}{3L}} \frac{4\pi}{L} \sin \frac{4\pi x}{L}
\]

\[
\frac{d^2\psi}{dx^2} = \sqrt{\frac{2}{3L}} \frac{16\pi^2}{L^2} \cos \frac{4\pi x}{L}
\]
\[ \langle p \rangle = \int_0^L \mathcal{P}_0 \psi^* \psi \, dx = \]
\[ = \int_0^L \sqrt{2 \frac{\sqrt{3L}}{2}} \left( 1 - \cos \frac{\pi L}{2} \right) \left( -i \hbar \right) \frac{d}{dx} \left( \frac{2}{3L} \left( 1 - \cos \frac{\pi y}{2} \right) \right) \, dx \]
\[ = \frac{2}{3L} \left( -i \hbar \right) \int_0^L \left( 1 - \cos \frac{\pi y}{2} \right) \frac{4\pi}{z} \sin \frac{\pi y}{z} \, dx \]
\[ = \frac{2}{3L} \left( -i \hbar \right) \left( \frac{4\pi}{z} \right) \int_0^L \left( \sin \frac{\pi y}{z} - \cos \frac{\pi y}{z} \sin \frac{\pi y}{z} \right) \, dx \]
\[ = -i \hbar \frac{8\pi}{3L} \left[ \int_0^L \left( \cos \frac{\pi y}{z} - \frac{1}{2} \sin^2 \frac{\pi y}{z} \right) \, dx \right] \]
\[ = i \hbar \frac{8\pi}{3L} \left[ \frac{1}{4\pi} \left( 1 - 1 \right) + \frac{1}{8\pi} \left( 0 - 0 \right) \right] \]
\[ \langle p \rangle = 0 \]

As expected for infinite square well!
\[ \mathcal{E} = \int_0^L 4x \, \varepsilon_0 \varepsilon_r \, dx = \int_0^L \left( \frac{2}{3L} \left( 1 - \cos \left( \frac{4\pi x}{L} \right) \right) \right) \frac{2}{3L} \frac{16\pi^2 \hbar^2}{L^2} \cos \frac{4\pi x}{L} \, dx \]

\[ = -\frac{2}{3L} \frac{16\pi^2 \hbar^2}{L^2} \int_0^L \left( \cos \frac{4\pi x}{L} - \cos^2 \frac{4\pi x}{L} \right) \, dx \]

\[ = -\frac{16\pi^2 \hbar^2}{3L^3 m} \left[ \cos \frac{4\pi x}{L} - \frac{1}{2} - \frac{L}{16\pi^2} \operatorname{sin} \frac{8\pi x}{L} \right]_0^L \]

\[ = -\frac{16\pi^2 \hbar^2}{3L^3 m} \left( -\frac{L}{2} \right) - \frac{8\pi^2 \hbar^2}{3mL^2} \]
This mixed state has $\langle E \rangle$ a little bit more than $E$ for $n=2$. We’ll learn later that a measurement of $E$ will never yield $E_2$!
\[ \Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} \]

\[ \langle p^2 \rangle = 2m \langle K.E. \rangle \]

For infinite square well \( E = K.E. \).

\[ \langle p^2 \rangle = 2 \hbar \cdot \frac{8 \pi^2 \hbar^2}{3 \hbar L^2} = \frac{16 \pi^2 \hbar^2}{3 L^2} \]

\[ \Delta p = \sqrt{\frac{16 \pi^2 \hbar^2}{3 L^2} - 0} \]

\[ \Delta p = \frac{4 \pi \hbar}{\sqrt{3} L} \]

(good practice to verify that the units are correct—can catch mistakes this way)
Example 4: Inf. Well + Step

- A particle of mass $m$ is bound in a potential well defined by:
  
  \[ V(x) = \infty \text{ for } x < 0 \]
  
  \[ V(x) = 0 \text{ for } 0 \leq x < L \]
  
  \[ V(x) = V_0 \text{ for } L \leq x < 2L \]
  
  and \[ V(x) = \infty \text{ for } x \geq 2L \]

- The energy $E$ of the particle satisfies $E > V_0$. 
(a) Write down the general solution for the eigenfunctions in the regions $0 \leq x < L$ and $L \leq x < 2L$. 

\[-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V\psi = \varepsilon \psi\]

In region (1):

\[\psi_1(x) = A \sin \kappa_1 x + B \cos \kappa_1 x\]

\[\kappa_1 = \sqrt{\frac{2mE}{\hbar^2}}\]

But $\psi_1(0) = 0 \Rightarrow B = 0$

$\Rightarrow \psi_1(x) = A \sin \kappa_1 x$

In region (2):

\[\psi_2(x) = C \sin \kappa_2 x + D \cos \kappa_2 x\]

\[\kappa_2 = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}\]

But $\psi_2(2L) = 0$. 
The algebra will be reduced if we write the general solution so that the boundary condition $\psi(2L) = 0$ is automatically satisfied:

$$
\psi_2(x) = C \sin \frac{x}{2L} (x-2L)
$$

Continuity at $x=L$ of $\psi$: $A \sin \frac{x}{L} L = C \sin \left( \frac{L}{2L} \right) (L-2L)$

$$
\Rightarrow A \sin \frac{x}{L} L = -C \sin \frac{L}{2L}
$$

$$
\Rightarrow C = -A \frac{\sin \frac{x}{L} L}{\sin \frac{L}{2L}}
$$

So, full solution is:

$$
\psi_1(x) = A \sin \frac{x}{L} L
$$

$$
\psi_2(x) = -A \frac{\sin \frac{x}{L} L}{\sin \frac{L}{2L}} \sin \frac{x}{2L} (x-2L)
$$
(b) Obtain an equation that implicitly expresses the quantization of energy. This equation must NOT contain any arbitrary constants.

\[
\begin{align*}
\text{Continuity at } x = L & \text{ of } \frac{dy}{dx}: \\
\kappa_1 A \cos k_1 L &= \kappa_2 C \cos k_2 (L - 2L) \\
\Rightarrow \kappa_1 A \cos k_1 L &= \kappa_2 C \cos k_2 L
\end{align*}
\]

Divide equation (1) by equation above:

\[
\frac{A \sin k_1 L}{\kappa_1 A \cos k_1 L} = -\frac{C \sin k_2 L}{\kappa_2 C \cos k_2 L}
\]

\[
\Rightarrow \frac{\tan k_1 L}{k_1} = -\frac{\tan k_2 L}{k_2}
\]

\[
\frac{\tan k_1 L}{\tan k_2 L} = -\frac{k_1}{k_2} = -\sqrt{\frac{E}{E-V_0}}
\]

This represents energy quantization.
Summary/Announcements

- Good luck on Midterm exam on Wed.
- Homework #6 due next Monday, Oct 28