Solution to HW#6

1. Reed: Chapter 5

Problem 5-3

Working from equations (5.3.1), (5.3.6), (5.3.14), and the normalization condition, develop explicitly the expression for the $n = 2$ harmonic oscillator wavefunction. Compare to Table 5.1.

The relevant expressions are

$$\psi(\xi) = H(\xi)e^{-\xi^2/2},$$

$$H(\xi) = \sum_{n=0}^{\infty} a_n \xi^n,$$

and

$$a_{n+2} = \frac{(2n+1-\lambda)}{(n+1)(n+2)} a_n.$$

For $n = 2$, equation (5.3.20), $\lambda = (2n+1)$, gives $\lambda = 5$. The recursion relation then appears as

$$a_{n+2} = \frac{(2n+1-5)}{(n+1)(n+2)} a_n.$$

For $n = 2$, we have only even-powered terms in the series: $n = 0$ and 2. Hence the recursion relation reduces to $a_2 = -2a_0$, and the wavefunction appears as

$$\psi_2(\xi) = a_0(1 - 2\xi^2)e^{-\xi^2/2}.$$

Normalization demands

$$\int_{-\infty}^{\infty} |\psi_2(x)|^2 dx = 1 \implies \frac{1}{\alpha} \int_{-\infty}^{\infty} |\psi_2(\xi)|^2 d\xi = 1,$$

where we have used the fact that $\xi = \alpha x$, hence $d\xi = \alpha dx$. Carrying out the normalization gives
\[
\frac{a_0^2}{\alpha} \int_{-\infty}^{\infty} (1 - 2\xi^2)^2 e^{-\xi^2/2} d\xi = 1,
\]
or
\[
\frac{a_0^2}{\alpha} \left\{ \int_{-\infty}^{\infty} e^{-\xi^2/2} d\xi - 4 \int_{-\infty}^{\infty} \xi^2 e^{-\xi^2/2} d\xi + 4 \int_{-\infty}^{\infty} \xi^4 e^{-\xi^2/2} d\xi \right\} = 1.
\]

All integrals are symmetric about \( x = 0 \). Changing the lower limit to zero introduces a leading factor of 2; the relevant integrals appear in Appendix C:

\[
2 \frac{a_0^2}{\alpha} \left\{ \frac{\sqrt{\pi}}{2} - 4 \frac{\sqrt{\pi}}{4} + 4 \frac{3\sqrt{\pi}}{8} \right\} = 1,
\]
or
\[
a_0 = \frac{\sqrt{\alpha}}{\sqrt{2} \pi^{1/4}}.
\]

The wavefunction then appears as
\[
\psi_2(\xi) = \frac{\sqrt{\alpha}}{\sqrt{2} \pi^{1/4}} (1 - 2\xi^2) e^{-\xi^2/2} = \frac{\sqrt{\alpha}}{2 \sqrt{2} \pi^{1/4}} (4\xi^2 - 2) e^{-\xi^2/2},
\]
where we suppress a negative sign in the final result since only \( \psi^2 \) will be of physical value. That the leading constant is correct can be seen by comparing to equation (5.4.2):

\[
A_2 = \sqrt{\frac{\alpha}{\sqrt{\pi} 2^2 2!}} = \frac{\sqrt{\alpha}}{2 \sqrt{2} \pi^{1/4}}.
\]
2. Reed: Chapter 5

Problem 5-5

Verify by explicit calculation that the $n = 0$ and $n = 2$ harmonic oscillator wavefunctions are orthonormal.

The wavefunctions are

$$\psi_0(x) = \frac{\sqrt{\alpha}}{\pi^{1/4}} e^{-\alpha^2 x^2/2}$$

and

$$\psi_2(x) = \sqrt{\frac{\alpha}{8\sqrt{\pi}}} (4\alpha^2 x^2 - 2) e^{-\alpha^2 x^2/2}.$$  

Hence

$$\int_{-\infty}^{\infty} \psi_0(x) \psi_2(x) dx = \frac{\alpha}{\sqrt{8\pi}} \int_{-\infty}^{\infty} (4\alpha^2 x^2 - 2)e^{-\alpha^2 x^2} dx$$

$$= \frac{8\alpha^3}{\sqrt{8\pi}} \int_{0}^{\infty} x^2 e^{-\alpha^2 x^2} dx - \frac{4\alpha}{\sqrt{8\pi}} \int_{0}^{\infty} e^{-\alpha^2 x^2} dx$$

$$= \frac{8\alpha^3}{\sqrt{8\pi}} \left( \frac{\sqrt{\pi}}{4\alpha^3} \right) - \frac{4\alpha}{\sqrt{8\pi}} \left( \frac{\sqrt{\pi}}{2\alpha} \right) = 0.$$
3. Reed: Chapter 5

Problem 5-6

Consider a particle of mass $m$ in the first excited state ($n=1$) of a harmonic oscillator potential. Compute the probability of finding the particle outside the classically allowed region.

Here we have

$$\psi_1(\xi) = \frac{\alpha}{\sqrt{2\sqrt{\pi}}} (2\xi) e^{-\xi^2/2}.$$

The probability of finding the particle *within* the classically allowed region is given by

$$P_{in} = \int_{-\xi_{turn}}^{\xi_{turn}} \psi_1^*(x)\psi_1(x)dx = \frac{1}{\alpha} \int_{-\xi_{turn}}^{\xi_{turn}} \psi_1(\xi)\psi_1(\xi)d\xi.$$ 

Where we have used $\xi = \alpha x$. Given that the turning points of the motion in the $n = 1$ case are $\xi_{turn} = \pm \sqrt{3}$ (see problem 5-8 below) and that the integral is symmetric about $\xi = 0$, we have

$$P_{in} = \frac{4}{\sqrt{\pi}} \int_0^{\sqrt{3}} \xi^2 e^{-\xi^2} d\xi.$$ 

The integral itself (apart from the factor of $4/\sqrt{\pi}$) evaluates to 0.393657. This gives $P_{in} = 0.8884$, or $P_{out} = 0.1116$.

(We also did this in class.)
4. **Reed: Chapter 5**

**Problem 5-16**

Show that application of the lowering operator $A^-$ to the $n = 3$ harmonic-oscillator wavefunction leads to the result predicted by equation (5.5.22).

The wavefunction is

$$
\psi_3(\xi) = A_3 \left(8\xi^3 - 12\xi\right) e^{-\xi^2/2}
$$

and the operator is

$$
A^- = \frac{t}{\sqrt{2}} \left( -\frac{d}{d\xi} - \xi \right).
$$

Now,

$$
\frac{d\psi_3}{d\xi} = A_3 \left(-8\xi^4 + 36\xi^2 - 12\right)e^{-\xi^2/2}.
$$

After a few lines of algebra, one finds

$$
A^-\psi_3 = -\frac{6t}{\sqrt{2}} A_3 \left(4\xi^2 - 2\right)e^{-\xi^2/2}.
$$

The $\xi$-dependence here is exactly that for the $n = 2$ state (Table 5.1). From equation (5.4.2), $A_3 = A_2/\sqrt{6}$, giving $A^-\psi_3 = -t\sqrt{3}\psi_2$, consistent with equation (5.5.22).
5. (a) 

\[ \langle x \rangle = \int \psi_n^* (x) \psi_n(x) \, dx \]

\[ = \frac{-i}{\sqrt{2}\alpha} \int \psi_n^* [A^+ - A^-] \psi_n \, dx \]

\[ = -\frac{i}{\sqrt{2}\alpha} \int \psi_n^* [A^+ \psi_n - A^- \psi_n] \, dx \]

\[ = -\frac{i}{\sqrt{2}\alpha} \left[ i\sqrt{n+1} \int \psi_n^* \psi_{n+1} \, dx + i\sqrt{n} \int \psi_n^* \psi_{n-1} \, dx \right] \]

Recall orthogonality:

\[ \int \psi_k^* \psi_n \, dx = 0 \quad \text{for } k \neq n \]

So,

\[ \int \psi_n^* \psi_{n+1} \, dx = 0 \quad \text{and} \quad \int \psi_n^* \psi_{n-1} \, dx = 0 \]

\[ \langle x \rangle = 0 + 0 = 0 \]

\[ \checkmark \]
(b) Before doing problem 5-17, recall that in class we found the following:

\[ \langle p \rangle = \int \psi_n^* (p_{op} \psi_n) \, dx \]

\[ = \frac{\alpha \hbar}{\sqrt{2}} \int \psi_n^* \left[ A^+ + A^- \right] \psi_n \, dx \]

\[ = \frac{\alpha \hbar}{\sqrt{2}} \int \psi_n^* (A^+ \psi_n) - \int \psi_n^* (A^- \psi_n) \, dx \]

\[ = \frac{\alpha \hbar}{\sqrt{2}} \left[ i \sqrt{\hbar \alpha} \int \psi_n^* \psi_{n+1} \, dx + i \sqrt{\hbar \alpha} \int \psi_n^* \psi_{n-1} \, dx \right] \]

\[ \therefore \langle p \rangle = 0 \quad \checkmark \]

\[ \langle x^2 \rangle = \int \psi_n^* (x_{\text{op}}^2 \psi_n) \, dx \]

\[ = \frac{i^2}{2 \alpha^2} \int \psi_n^* \left[ (A^+ - A^-)(A^+ \psi_n - A^- \psi_n) \right] \, dx \]

\[ = -\frac{1}{2 \alpha^2} \int \psi_n^* \left[ A^+ A^+ \psi_n - A^+ A^- \psi_n - A^- A^+ \psi_n + A^- A^- \psi_n \right] \, dx \]

\[ = \frac{\alpha \psi_{n+2}}{\psi_n \psi_{n+2}} \]

\[ = 0 \]

\[ = 0 \]
Just like in Reed 5.6 and what we did in class!

Use this result to do problem 5-17....
Problem 5-17
Following the approach that led to equation (5.5.30), use the raising and lowering operators to show that \( \langle p^2 \rangle = \alpha^2 \hbar^2 (n + 1/2) \) for the \( n \)th harmonic-oscillator state, and hence verify that \( \Delta x \Delta p \) is as given in problem 5-10.

The momentum operator is

\[
p_{\psi} = \frac{\alpha \hbar}{\sqrt{2}} (A^+ + A^-)
\]

Operate this on harmonic-oscillator state \( \psi \):

\[
\langle p^2 \rangle = \int \psi_n^*(p_{\psi} \psi_n) dx = \frac{\alpha^2 \hbar^2}{2} \int \psi_n [(A^+ + A^-)(A^- \psi_n + A^+ \psi_n)] dx
\]

\[
= \frac{\alpha^2 \hbar^2}{2} \int \psi_n (A^+ A^- \psi_n + A^- A^+ \psi_n + A^- A^- \psi_n + A^- A^+ \psi_n) dx.
\]

By orthogonality, only the second and third terms make non-zero contributions:

\[
\langle p^2 \rangle = \frac{\alpha^2 \hbar^2}{2} \int \psi_n (A^- A^+ \psi_n + A^- A^- \psi_n) dx.
\]

Just as in the calculation of \( \langle x^2 \rangle \), the bracketed term is, but for a constant, the Hamiltonian operator:

\[
H = \frac{\hbar \omega}{2} (A^- A^+ + A^+ A^-),
\]

hence

\[
\langle p^2 \rangle = \frac{\alpha^2 \hbar^2}{2} \frac{2}{\hbar \omega} \int \psi_n (H \psi_n) dx = \frac{\alpha^2 \hbar}{\omega} E_n = \alpha^2 \hbar^2 (n + 1/2),
\]

exactly as determined in Problem 5-10; \( \Delta x \Delta p \) follows directly.

Finally,

\[
\Delta x \Delta p = \sqrt{\langle x^2 \rangle \langle p^2 \rangle} = (n + 1/2) \hbar.
\]

(since \( \langle x \rangle = \langle p \rangle = 0 \))