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**Problem 3-3**

Using classical arguments, derive an expression for the speed $v$ of a particle in a one-dimensional infinite potential well. Apply your result to the case of an electron in a well with $L = 1\,\text{Å}$. For what value of $n$ does $v$ exceed the speed of light? To what energy does this correspond?

By equating the energy of a particle in state $n$ of an infinite well to the classical kinetic energy $mv^2/2$, 

$$E = \frac{n^2 \hbar^2}{8mL^2} = \frac{mv^2}{2},$$

we find 

$$v = \frac{n\hbar}{2mL}.$$

For an electron in a well with $L = 1\,\text{Å}$, this gives $v = (3.637 \times 10^6 \,\text{n}) \,\text{m/sec}$; $v$ exceeds $c$ for $n \sim 82$, which corresponds to an energy of 259 kV.
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Problem 3-5

A particle in some potential is described by the wavefunction $\psi(x) = Axe^{-kx}$ $(0 \leq x \leq \infty; k > 0$; see Problem 2-5.) If $k = 0.5\text{Å}^{-1}$, what is the probability of finding the particle between $x = 2.0$ and 2.1Å?

From Problem 2.5 the normalization of this wavefunction is given by $A^2 = 4k^3$. Hence

$$\Psi(x) = Axe^{-kx} \text{ from } x > 0$$

$$P = \int_{x_1}^{x_2} \Psi^* \Psi \, dx = \int_{x_1}^{x_2} A^2 x^2 e^{-2kx} \, dx$$

Using integrals from Appendix C:

$$P = \frac{A^2}{4k^3} \left( -e^{-2kx}(2k^2x^2 + 2kx + 1) \right) \bigg|_{x_1}^{x_2}$$

Plugging in expression for $A$ above and values: $k=0.5 \text{ Å}$, $x_1 = 2.0 \text{ Å}$ and $x_2 = 2.1 \text{ Å}$ one gets:

$$P = 0.0271 \text{ or } 2.71\%$$

$$P(x, x + \Delta x) = \psi^2(x) \Delta x = \left( A^2 x^2 e^{-2kx} \right) \Delta x = \left( 4k^3 x^2 e^{-2kx} \right) \Delta x$$

$$= \left\{ 4(0.5\text{Å}^{-1})^3 (2\text{Å})^2 \exp\left[-2(0.5\text{Å}^{-1})(2.0\text{Å})\right] \right\} (0.1\text{Å}) = 0.0271.$$
Problem 3-9

A proton is moving within a nuclear potential of depth 25 MeV and full width \(2L = 10^{14}\) meters. If the potential can be modeled as a finite rectangular well, how many energy states are available to the proton?

The number of states is given by

\[
N(K) = 1 + \left[ \frac{2K}{\pi} \right] = 1 + \left[ \frac{2}{\pi} \sqrt{\frac{2mV_oL^2}{h^2}} \right],
\]

where the square brackets designate the largest integer less than or equal to the argument within. With \(V_o = 25\) MeV \(= 4.00 \times 10^{-12}\) J, \(L = 0.5 \times 10^{-14}\) m, and a proton \((1.67 \times 10^{-27}\) kg),

\[
N(K) = 1 + \left[ \frac{2}{\pi} \sqrt{\frac{2(1.67 \times 10^{-27})(4.00 \times 10^{-12})(0.5 \times 10^{-14})^2}{(1.055 \times 10^{-34})^2}} \right]
\]

\[= 1 + [3.487] = 4.\]

Only four energy states are available to the proton in this situation.
Problem 3-11

Consider a particle of mass $m$ moving in the “semi-infinite” potential well illustrated below. Set up and solve the SWE for this system; assume $E < V_o$. Applying the appropriate boundary conditions at $x = 0$ and $x = L$, derive an expression for the permissible energy eigenvalues. With $V_o = 10 \text{ eV}$, $L = 5\text{ Å}$ and $m = m_{\text{electron}}$, compute the values of the energy (in eV) for the two lowest bound states.
Define region 1 as that for which $x < 0$, region 2 as $0 \leq x \leq L$, and region 3 as $x > L$. Since $V = \infty$ in region 1, $\psi = 0$ there. We restrict attention to bound states, that is, $E < V_0$. In regions 2 and 3 the solutions of the Schrödinger equation take the usual sinusoidal and exponential forms:

\[ \psi_2 = A \cos(k_2 x) + B \sin(k_2 x), \quad k_2 = \sqrt{2mE/\hbar^2} \]

and

\[ \psi_3 = C \exp(k_3 x) + D \exp(-k_3 x), \quad k_3 = \sqrt{2m(V_0 - E)/\hbar^2}. \]

At $x = 0$, we must have $\psi_2 = 0$, which forces $A = 0$. As $x \to \infty$, we must have $\psi_3 \to 0$, which forces $C = 0$. Demanding the continuity of $\psi_2$ and $\psi_3$ and their first derivatives at $x = L$ leads to

\[ B \sin(k_2 L) = D \exp(-k_3 L) \]

and

\[ k_2 B \cos(k_2 L) = -k_3 D \exp(-k_3 L). \]

Dividing the first of these conditions by the second (or vice-versa) gives the energy eigenvalue condition

\[ -k_3 \tan(k_2 L) = k_2. \]

Defining dimensionless variables $\xi = k_2 L$ and $\eta = k_3 L$, this condition can be expressed as

\[ -\eta \tan(\xi) = \xi, \]

$\xi$ and $\eta$ also satisfy

\[ \xi^2 + \eta^2 = \frac{2mV_0 L^2}{\hbar^2} = K^2. \]

For the values specified, $K^2 = 65.561$. This result can be used to eliminate $\eta$ (or $\xi$) from the eigenvalue equation, leaving

\[ -\sqrt{65.561 - \xi^2} \tan(\xi) = \xi. \]

The first two roots of this equation are $\xi_1 = 2.78983$ and $\xi_2 = 5.53117$; these correspond to energies of 1.187 and 4.667 eV, respectively.
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Problem 3-12

Derive an expression analogous to equation (3.4.11) for the odd-parity states of the finite rectangular well.

For the odd-parity states of the finite rectangular well we have \((F = -C)\)

\[
\begin{align*}
\psi_1(x) &= C \, e^{k_1 x} & (x < -L) \\
\psi_2(x) &= A \, \sin(k_2 x) & (-L \leq x \leq L) \\
\psi_3(x) &= -C \, e^{-k_1 x} & (x > L).
\end{align*}
\]

The boundary conditions demand

\[-A \sin \frac{\xi}{C} = C e^{-\eta}\]

and

\[k_2 A \cos \frac{\xi}{C} = k_1 C e^{-\eta} .\]

(We do not actually use this latter condition). From the first of these we can write

\[C^2 = A^2 \sin^2 \frac{\xi}{C} \, e^{2\eta} .\]

Normalizing the wavefunction demands

\[
\int_{-\infty}^{-L} \psi_1^* \psi_1 \, dx + \int_{-L}^{L} \psi_2^* \psi_2 \, dx + \int_{L}^{\infty} \psi_3^* \psi_3 \, dx = 1 ,
\]

or

\[C^2 \int_{-\infty}^{-L} e^{2k_1 x} \, dx + A^2 \int_{-L}^{L} \sin^2(k_2 x) \, dx + C^2 \int_{L}^{\infty} e^{-2k_1 x} \, dx = 1 .\]

The first and third integrals are identical. The normalization condition reduces to

\[\frac{C^2}{k_1} e^{-2\eta} + A^2 \left[ L - \frac{\sin(2\xi)}{2k_2} \right] = 1 .\]

Eliminating \(A^2\) via the boundary-condition information and solving for \(C^2\) gives

\[C^2 = \frac{\eta \xi \, e^{2\eta}}{L \left[ \xi + \eta \xi \csc^2 \frac{\xi}{C} - \eta \cot \frac{\xi}{C} \right]} .\]

The probability of finding the particle outside the confines of the well is then given by
\[ p_{\text{out}}^{\text{odd}} = \int_{-L}^{L} \psi_1^* \psi_1 \, dx + \int_{-L}^{L} \psi_3^* \psi_3 \, dx = \frac{C^2 e^{-2\eta}}{k_1} = \frac{\eta \xi e^{2\eta}}{L \left[ \xi + \eta \xi \csc^2 \xi - \eta \cot \xi \right]} \left( \frac{e^{-2\eta}}{k_1} \right), \]

or, since \( k_1 L = \eta \),

\[ p_{\text{out}}^{\text{odd}} = \frac{\xi}{\left[ \xi + \eta \xi \csc^2 \xi - \eta \cot \xi \right]}. \]