Solutions to HW#2

1. Reed: Chapter 2

**Problem 2-1**

Verify that if one has a number of independent solutions \( y_1, y_2, \ldots y_n \) of the classical wave equation (all of pattern speed \( v \)), then a linear sum of them (equation 2.1.10) is also a solution.

Each independent solution \( y_m(x) \) must satisfy

\[
\frac{\partial^2 y_m}{\partial t^2} = v^2 \frac{\partial^2 y_m}{\partial x^2}.
\]

A linear sum of \( y_n \)'s appears as

\[
y = a_1 y_1 + a_2 y_2 + \ldots + a_n y_n = \sum a_n y_n.
\]

Differentiating this linear sum and using the first equation above gives

\[
\frac{\partial^2 y}{\partial x^2} = \sum a_n \frac{\partial^2 y_n}{\partial x^2} = v^2 \sum a_n \frac{\partial^2 y_n}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial t^2}.
\]
Problem 2-4

Suppose that the solution to the Schrödinger equation for some potential gives rise to three wavefunctions, \( \psi_1(x) \), \( \psi_2(x) \), and \( \psi_3(x) \) of the forms and domains

\[
\begin{align*}
\psi_1(x) &= A e^{kx} & (-\infty \leq x \leq 0) \\
\psi_2(x) &= Bx^2 + Cx + D & (0 \leq x \leq L) \\
\psi_3(x) &= 0 & (x \geq L).
\end{align*}
\]

What values must \( B, C, \) and \( D \) take in terms of \( A \) in order that these solutions satisfy the continuity conditions discussed in Section 2.4? In Chapter 3 we will see that a solution of the form \( \psi_3(x) = 0 \) requires the presence of an infinitely discontinuous potential at \( x = L \); consequently, you do not need to apply the continuity condition on \( d\psi/dx \) there. Derive also an expression that \( A, B, C, D, k \) and \( L \) must satisfy in order that the overall \( (-\infty \leq x \leq +\infty) \) solution be normalized.

The required conditions are

\[
\psi_1(0) = \psi_2(0),
\]

\[
\left. \frac{d\psi_1}{dx} \right|_0 = \left. \frac{d\psi_2}{dx} \right|_0,
\]

and

\[
\psi_2(L) = \psi_3(L).
\]

Condition (1) gives

\[
D = A.
\]

Taking the derivatives of \( \psi_1 \) and \( \psi_2 \) and then substituting \( x = 0 \) renders condition (2) as

\[
C = kA.
\]

Condition (3) appears as

\[
BL^2 + CL + D = 0.
\]

Results (4) and (5) in (6) gives \( B \) explicitly in terms of \( A \):

\[
B = -\frac{A}{L^2}(1 + kL).
\]
To normalize the solution, we must have

$$\int_{-\infty}^{0} \psi_1^2 dx + \int_{0}^{L} \psi_2^2 dx + \int_{L}^{\infty} \psi_3^2 dx = 1.$$ 

The integral over $\psi_3$ gives zero since $\psi_3 = 0$ over its domain of applicability. The other two integrals give

$$\int_{-\infty}^{0} \psi_1^2 dx = A^2 \int_{-\infty}^{0} e^{2kx} dx = \frac{A^2}{2k} \left[ e^{2kx} \right]_{-\infty}^{0} = \frac{A^2}{2k},$$

and

$$\int_{0}^{L} \psi_2^2 dx = \int_{0}^{L} (Bx^2 + Cx + D)^2 dx$$

$$= \int_{0}^{L} [B^2x^4 + (2BC)x^3 + (2BD + C^2)x^2 + (2CD)x + D^2] dx$$

$$= \frac{1}{5} B^2 L^5 + \frac{1}{2} (BC) L^4 + \frac{1}{3} (2BD + C^2) L^3 + (CD) L^2 + D^2 L.$$ 

The normalization condition is then

$$\frac{A^2}{2k} + \frac{1}{5} B^2 L^5 + \frac{1}{2} (BC) L^4 + \frac{1}{3} (2BD + C^2) L^3 + (CD) L^2 + D^2 L = 1.$$ 

Results (4)-(6) could be substituted here to render the normalization entirely in terms of $A$, $k$, and $L$, but the final expression would not be particularly compact.
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Problem 2-5

Given the wavefunction $\psi(x) = Axe^{-kx}$ ($0 \leq x \leq \infty$; $k > 0$), what value must $A$ take in terms of $k$ in order that $\psi$ be normalized? See Appendix C for the relevant integral.

We must have

$$\int_{0}^{\infty} \psi^2(x) \, dx = 1,$$

that is,

$$A^2 \int_{0}^{\infty} x^2 e^{-2kx} \, dx = 1.$$

This integral can be evaluated using the table of integrals in Appendix C; the result is

$$\frac{A^2}{4k^3} = 1,$$

or $A = 2k^{3/2}$. 
Problem 2-6

Suppose that the wavefunction in problem 2-5 is known to be a solution of the Schrödinger equation for some energy $E$. What is the corresponding potential function $V(x)$?

We have

$$\psi(x) = Axe^{-kx}.$$  

Hence

$$\frac{d\psi}{dx} = Ae^{-kx} - kAxe^{-kx},$$

and

$$\frac{d^2\psi}{dx^2} = (-2k + k^2x)Ae^{-kx}.$$  

The Schrödinger equation is

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V(x)\psi = E\psi.$$  

Substituting the second derivative gives

$$-\frac{\hbar^2}{2m}(-2k + k^2x)Ae^{-kx} + V(x)Axe^{-kx} = EAxe^{-kx}.$$  

Canceling the common factor of $Ae^{-kx}$ and dividing through by $x$ leaves

$$V(x) = E + \frac{\hbar^2k^2}{2m} - \frac{\hbar^2}{mx}.$$  

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**Problem 2-7**

The wavefunction $\psi(x) = Ae^{-kx^2}$ ($-\infty \leq x \leq \infty$; $k > 0$) is known to be a solution of the Schrödinger equation for some energy $E$. What is the corresponding potential function $V(x)$? We will explore a potential like this in Chapter 5.

We need the second derivative of $\psi$:

$$\psi(x) = Ae^{-kx^2},$$

so

$$\frac{d\psi}{dx} = -(2kA)e^{-kx^2},$$

hence

$$\frac{d^2\psi}{dx^2} = -(2kA)e^{-kx^2} + (4k^2A)x^2e^{-kx^2}.$$

The Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi.$$

Substituting the second derivative gives

$$-\frac{\hbar^2}{2m}[-(2k) + (4k^2)x^2]Ae^{-kx^2} + V(x)Ae^{-kx^2} = EAe^{-kx^2}.$$

Canceling the common factor of $Ae^{-kx^2}$ leaves

$$V(x) = E + \frac{\hbar^2}{2m}(4k^2x^2 - 2k).$$

This is a more elaborate version of the harmonic oscillator potential of Chapter 5.