Solution to HW#1

1. Problem 1-3

Equation (1.2.13) gives the amount of radiant energy $E_d dv$ between frequencies $\nu$ and $\nu + dv$ within a blackbody cavity. According to Einstein, the energy of an individual photon of frequency $\nu$ is $E = h\nu$. If the radiant energy is presumed to be comprised of individual photons, then the number of photons with frequencies between $\nu$ and $\nu + dv$ in the cavity would be given by $n_d dv = E_d dv / h\nu$. The total number of photons within the cavity will then be given by the integral of this expression for $\nu = 0$ to $\infty$. Given that

$$\int_0^\infty \frac{x^2 dx}{(e^x - 1)} \sim 2.404,$$

determine the density of photons in a blackbody cavity at $T = 300$ K, about room temperature. Intergalactic space has a temperature of about 2.7 K; what is the density of photons in that environment?

Here we have

$$n_d dv = \frac{E_d dv}{h\nu} = \left(\frac{1}{h\nu}\right)^2 \frac{8\pi hV}{c^3} \frac{\nu^3}{(e^{h\nu/kT} - 1)} dv = \frac{8\pi V}{c^3} \frac{\nu^2}{(e^{h\nu/kT} - 1)} dv.$$

Integrating over all frequencies to get the total number of photons $N$ in the cavity gives

$$N = \int_0^\infty n_d dv = \frac{8\pi V}{c^3} \int_0^\infty \frac{\nu^2}{(e^{h\nu/kT} - 1)} dv.$$

Define $x = h\nu/kT$ to use the hint given:

$$N = \frac{8\pi V}{c^3} \left(\frac{kT}{h}\right)^3 \int_0^\infty \frac{x^2}{(e^x - 1)} dx = 2.404 \left(\frac{8\pi V}{c h}\right) \left(\frac{kT}{c h}\right)^3.$$

Inserting values for the physical constants (MKS units) gives

$$N = 2.404 \left(\frac{8\pi V}{c h}\right) \left(\frac{1.381 \times 10^{-23}}{(2.998 \times 10^8)(6.626 \times 10^{-34})}\right)^3 \left(T^3\right) = \left(2.030 \times 10^7\right)(VT^3).$$

For $V = 1$ m$^3$ and $T = 300$ K, $N = 5.48 \times 10^{14}$. $T = 2.7$ gives $4.00 \times 10^8$, or 400 per cubic centimeter.
2.

Problem 1-7

Derive an expression for the centripetal force experienced by the electron in Bohr orbit “n” as a function of n and physical constants. What is the value of the force when n = 1? Modern force-probe microscopes are capable of nano-Newton scale forces.

Equations (1.3.1) and (1.3.7) give

\[ F_c = \frac{e^2}{4\pi \varepsilon_0 r^2} = \frac{e^2}{4\pi \varepsilon_0} \left( \frac{\pi m_e e^2}{\varepsilon_0 h^2 n^2} \right) = \left( \frac{\pi m_e^2 e^4}{4\varepsilon_0^3 h^4} \right) \frac{1}{n^4}. \]

Substituting the appropriate numerical values gives

\[ F_c = \frac{8.233 \times 10^{-8} \text{ N}}{n^4}. \]

For n = 1, we have 82.3 nano-Newton, a result of the order of force-probe microscopy.

Actually, these days forces of the order of a few pico-Newton can now be routinely measured
Problem 1-8

An electron transits between two energy levels separated by energy $E$, emitting a photon of wavelength $\lambda$ in accordance with the Planck/Bohr relation $E = \frac{hc}{\lambda}$. (a) Show that if $\lambda$ is given in units of Ångstroms, then $E = 12398/\lambda$ electron-volts. (b) As a consequence of some perturbing effect, the separation of the energy levels is altered by an amount $dE$ electron-volts. Show that the corresponding change in the wavelength of the emitted photon is $d\lambda = -\lambda^2 dE/12398$ Ångstroms. If a transition leading to a photon of wavelength 4000Å is perturbed by $dE = 0.01$ eV, what is $d\lambda$?

(a) We have

$$E = \frac{hc}{\lambda}.$$ 

If $h$, $c$, and $\lambda$ are specified in MKS units, then $E$ must emerge in Joules. If $\lambda$ is given instead in Å (written as $\lambda_\text{Å}$), it can be converted to meters by multiplying by $10^{10}$, hence,

$$E = \frac{hc}{10^{10} \lambda_\text{Å}} = \frac{(6.6261 \times 10^{-34} \text{J} \cdot \text{sec})(2.9979 \times 10^8 \text{m/sec})}{10^{10} \lambda_\text{Å} (\text{m})} = \frac{1.9864 \times 10^{-15}}{\lambda_\text{Å}} \text{ Joules.}$$

Now, 1 Joule $= 1.6022 \times 10^{-19}$ eV, hence

$$E_{\text{eV}} = \frac{1.9864 \times 10^{-15}}{(1.6022 \times 10^{-19}) \lambda_\text{Å}} = \frac{12398}{\lambda_\text{Å}}.$$ 

(b) To examine $d\lambda$, take the derivative of $\lambda = hc/E$

$$d\lambda = -\frac{hc}{E^2} dE = -\left(\frac{hc}{E}\right) \frac{dE}{E} = -\lambda \frac{dE}{E}.$$ 

Notice that $dE/E$ is dimensionless if both $dE$ and $E$ are given in the same units. Demand that both $dE$ and $E$ are specified in eV. From part (a), then

$$\lambda = -\lambda \frac{dE_{\text{eV}}}{E_{\text{eV}}} = -\frac{\lambda \lambda_\text{Å}}{12398} dE_{\text{eV}}.$$ 

This would give $d\lambda$ in meters when $\lambda$ is given in meters (note that the two factors of $\lambda$ appearing here are in different units: meters and Å, respectively.) But, converting both $d\lambda$ and to Å would introduce factors of $10^{10}$ on both sides, which cancel each other out. Hence

$$d\lambda_\text{Å} = -\frac{\lambda_\text{Å}^2}{12398} dE_{\text{eV}}.$$ 

For $\lambda_\text{Å} = 4000$ and $dE = 0.01$, $d\lambda_\text{Å} = 12.91$Å.
Problem 1-19

Particles in motion have associated de Broglie wavelengths. As a rule, if one is within a few de
Broglie wavelengths of a particle then the particle loses its identity as a particle and acts more
like a wave, with the result that classical mechanics no longer suffices to describe collisions.
Rutherford investigated the scattering of $\alpha$-particles (helium nuclei) from stationary gold nuclei,
using classical mechanics to analyze the collisions. This question investigates the issue from a
wave-mechanics perspective. (a) Determine the de Broglie wavelength of an $\alpha$-particle of kinetic
energy 6 million electron-volts. (b) The electrical potential energy of charges $Q_1$ and $Q_2$
separated by distance $r$ is given by $U = \frac{Q_1 Q_2}{4 \pi \epsilon_0 r}$. By equating the initial kinetic energy of the
$\alpha$-particle to the electrical potential energy, find the distance of closest approach of the $\alpha$-particle
to the gold nucleus, and so render a judgment as to how well-justified Rutherford was in his use
of classical mechanics.

The de Broglie wavelength of a 6-MeV $\alpha$-particle ($m = 6.64 \times 10^{-27}$ kg) is

$$\lambda = \frac{\hbar}{\sqrt{2Km}} = \frac{6.626 \times 10^{-34} J \cdot \text{sec}}{\sqrt{2(6)(1.602 \times 10^{-19} J/\text{MeV})(6.64 \times 10^{-27} \text{kg})}} = 5.87 \times 10^{-15} \text{ m.}$$

Gold has $Z = 79$; the closest-approach distance is then

$$K = \frac{(2e)(79e)}{4 \pi \epsilon_0 r} \rightarrow r = \frac{158e^2}{4 \pi \epsilon_0 K} = 3.79 \times 10^{-14} \text{ m.}$$

The ratio of the closest-approach distance to $\lambda$ is about 6.47: not terribly large. Rutherford’s use
of classical mechanics was marginally justified.
Problem 1-20

In kinetic theory, atoms are modeled as point masses of mass \( m \), and the mean speed \( v \) of an atom in an environment at absolute temperature \( T \) is given by \( mv^2 = 3kT \) where \( k \) is Boltzmann’s constant. Derive an expression for the de Broglie wavelength for an atom of mass \( m \) at absolute temperature \( T \) in terms of \( h \), \( k \), \( T \), and \( m \). At room temperature, hydrogen atoms typically travel many thousands of Ångstroms between collisions; is it fair to treat them as effectively acting as point masses under such circumstances?

\[
E = mc^2 + K = \frac{mv^2}{2} = \frac{3}{2} kT
\]

non-relativistic so can safely use

\[
K = \frac{p^2}{2m}
\]

\[
p = \sqrt{2mK}
\]

\[
\lambda = \frac{\hbar}{p} = \frac{\hbar}{\sqrt{2mK}} = \frac{\hbar}{\sqrt{2m \cdot \frac{3}{2} kT}} = \frac{\hbar}{\sqrt{3m kT}}
\]

\[
T = 300 \text{ K} \quad m = 1u = 1.67 \times 10^{-24} \text{ kg} \quad (H \text{ atom})
\]

\[
k = 1.38 \times 10^{-23} \text{ J/K} \quad h = 6.63 \times 10^{-34} \text{ J.s}
\]

\[
\lambda = \frac{6.63 \times 10^{-34} \text{ J.s}}{\sqrt{3 \cdot (1.67 \times 10^{-24} \text{ kg}) \cdot (1.38 \times 10^{-23} \text{ J/K}) \cdot 300}}
\]

\[
= 1.5 \times 10^{-10} \text{ m}
\]

\[
\lambda = 1.5 \AA \quad \text{it is safe to regard the H atoms as point masses}
\]
6. 

**Problem 1-22**

An atom of mass $m$ is initially at rest with respect to an outside observer and has an electron in an excited state. The electron transits to a lower energy level, emitting a single photon of wavelength $\lambda$ in the process. The emitted photon will have momentum $p = h/\lambda$. To conserve momentum, the atom must recoil with some speed $v$. Derive an expression giving the ratio of the photon’s energy to the kinetic energy of the recoiling atom in terms of $\lambda$, $m$, $c$, and $h$. A classical analysis will suffice for the recoiling atom. Evaluate the ratio numerically for a hydrogen atom emitting a photon of wavelength 5000Å. Based on your result, do you think it was reasonable to neglect the energy involved with the recoiling atom in the derivation of the Bohr model? Along these lines, the momentum transferred to an atom upon its absorbing a photon has actually been measured: see Physics Today 2005;58(7):9.

The energy of the photon is $E = hc/\lambda$. From equation (1.4.1), the momentum of the photon is $p = h/\lambda$. Hence, for the recoiling atom, $mv = h/\lambda$, so $v = h/\lambda m$. The ratio is

$$\frac{E_{\text{photon}}}{E_{\text{recoil}}} = \frac{hc}{\lambda} \left(\frac{2}{mv^2}\right) = \frac{hc}{\lambda} \left(\frac{2}{m}\right) \left(\frac{\lambda m}{h}\right)^2 = \frac{2mc\lambda}{h}.$$

For a hydrogen atom and a photon of $\lambda = 5000Å$,

$$\frac{E_{\text{photon}}}{E_{\text{recoil}}} = \frac{2(1.67 \times 10^{-27})(2.998 \times 10^9)(5 \times 10^{-7})}{(6.626 \times 10^{-34})} = 7.56 \times 10^8.$$

The vast majority of the transition energy is carried off by the photon. It is safe to neglect the recoiling atom in the derivation of the Bohr model.