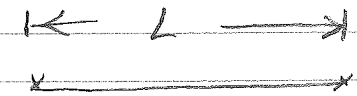
LECTURE 18

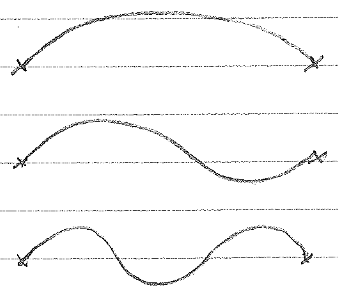
LAB ON HARMONICS.

FOURIER SERIES. - SPATIAL REPRESENTATION



STRING.

STANDING WAVES ON A STRING.



$\lambda_1 = 2L$ $\qquad k_1 = \frac{2\pi}{\lambda_1} = \frac{2\pi}{2L} = \frac{\pi}{L}$

$2\lambda_2 = 2L$ $\qquad k_2 = \frac{2\pi}{\lambda_2} = \frac{2\pi}{L}$

$3\lambda_3 = 2L$ $\qquad k_3 = \frac{2\pi}{\lambda_3} = 3\frac{\pi}{L}$

$n\lambda_n = 2L$ $\qquad k_n = n\frac{\pi}{L}$

FOR A PURE SINE WAVE AT $t=0$ WITH λ_1

$$y(x,0) = A \sin k_1 x \cos 0t = A \sin \frac{\pi}{L} x \cos(0t)$$

$$y(x,t) = A \sin \frac{\pi}{L} x \cos \omega_1 t \qquad \omega_1 = k_1 v$$

$$v = \sqrt{\frac{T}{M/L}}$$

IF WE HAVE A MIXTURE OF λ_1 AND λ_2

$$y(x,t)\Big|_{t=0} = A_1 \sin \frac{\pi}{L} x + A_2 \sin \frac{2\pi}{L} x$$

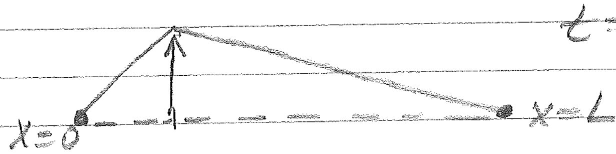
A_1 AMPLITUDE FOR k_1, A_2 AMPLITUDE FOR k_2.

TIME EVOLUTION WOULD BE

$$y(x,t) = A_1 \sin \frac{\pi}{L} x \cos \omega_1 t + A_2 \sin \frac{2\pi}{L} x \cos \omega_2 t$$

$$\omega_2 = k_2 v.$$

PLUCK A STRING. (GUITAR, PLUCKED VIOLIN,...)



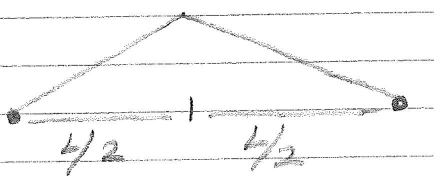
CAN WE EXPRESS THIS SHAPE AS A FUNCTION OF SINE'S

i.e. $y(x,t=0) = \sum A_n \sin k_n x \qquad k_n L = n\pi$

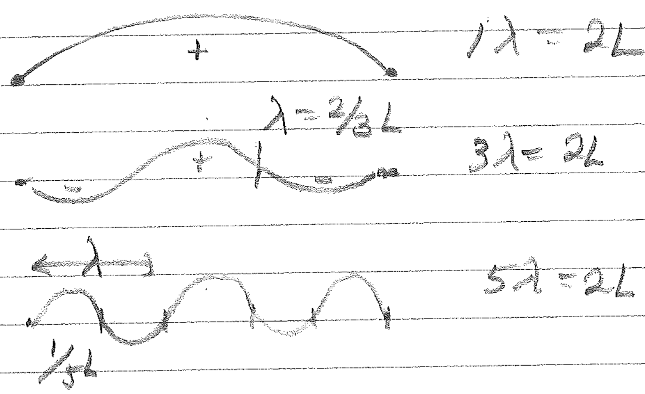
IF SO $y(x,t) = \sum A_n \sin k_n x \cos \omega_n t \qquad \omega_n = k_n v$

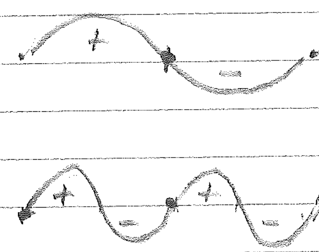
or $f_n \lambda_n = v$

SOME QUALITATIVE FEATURES. - PLUCK IN THE MIDDLE



EVEN HARMONICS NOT PRESENT



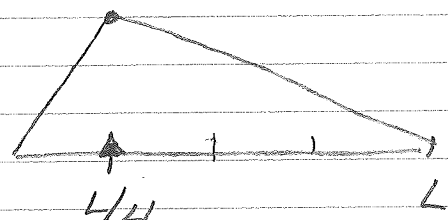


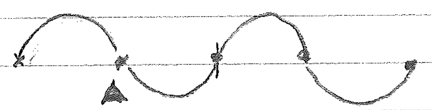
All ODD HARMONICS ARE PRESENT

THESE HAVE A NODE AT $x=L/2$, STRING HAS A MAXIMUM.

2ND HARMONIC IS ALS ASYMMETRIC

PLUCK AT $X = L/4$



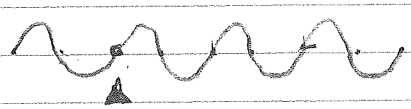


THIS HARMONIC HAS NODE AT $L/4$ AND IT IS MISSING.

$\lambda = L/2 \quad 4\lambda = 2L$

4^{th} HARMONIC. $\quad n\lambda = 2L$

n'th HARMONIC



$8\lambda = 2L$.

8^{th} HARMONIC IS ALSO MISSING

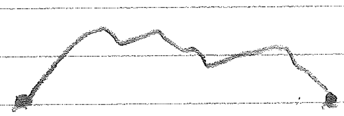
$\Rightarrow$ All OVERTONES OF 4^{th} HARMONIC ARE MISSING $\quad n = 4, 8, 12, 16, \ldots$ MISSING.

$$y = A_1 \sin k_1 x + A_2 \sin k_2 x + A_3 \sin k_3 x + 0 \sin k_4 x + \ldots + 0 \sin k_8 x \;\vdots$$

FOURIER SERIES:

A SHAPE $y(x,0)$ CAN BE REPRESENTED AS

$$y(x,0) = \sum A_n \sin k_n x + \underbrace{B_n \cos k_n x}_{\text{MORE GENERALLY}}$$



IF $y(x=0, t=0) = 0$ NO $\cos k_n x$ TERMS.

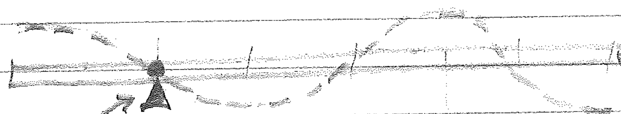
$$y(x, t=0) = \sum A_n \sin k_n x \qquad k_n L = n\pi$$

$$= \sum A_n \sin \frac{n\pi x}{L} \qquad \omega_n = 2\pi f_n = 2\pi n f_1$$

$$y(x,t) = \sum A_n \sin \frac{n\pi x}{L} \cos \omega_n t$$

NOTE IN SINGING RODS THE NODES DETERMINE THE HARMONICS

OPEN ENDS.

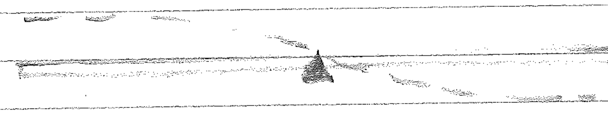


OPEN ENDS

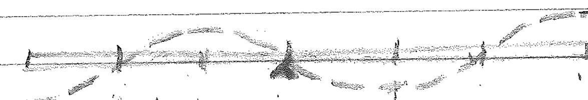
HAND ROD HERE

THIS POINT HAS TO BE A NODE & IT DETERMINES FUNDAMENTAL & ALL OTHER λ'S.

HOLD AT $x = L/2$



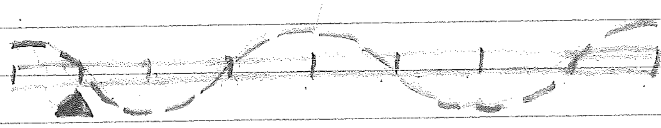
$1\lambda = 2L$ $\quad 1\lambda = 2L$



$\frac{1}{6}L$ $\quad \frac{2}{3}L$

$\lambda = \frac{2}{3}L$. $\quad 3\lambda = 2L$.

ODD HARMONICS AGAIN.



$\lambda = L/8$ $\quad \frac{2}{8}L \quad \frac{3L}{8} \quad \frac{4L}{8} \quad \frac{5L}{8} \quad \frac{6L}{8} \quad \frac{7L}{8} \quad \frac{8L}{8}$

$2\lambda = L$

$4\lambda = 2L$.

THIS IS FUNDAMENTAL

NEXT λ IS:

$12\lambda = 2L$

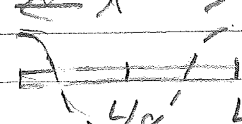
THEN $20\lambda = 2L$

ODD BUILT ON $4\lambda = 2L$

* NOTE NO $8\lambda = 2L$

$\lambda = \frac{1}{4}L$.

$\longleftarrow \lambda \longrightarrow$



$L/8 \quad L/4$

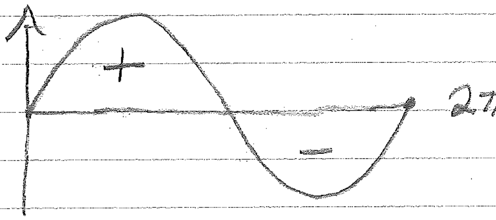
START WITH ANTI-NODE - DON'T GET A NODE AT $L/8$. IF $\lambda = \frac{1}{4}L$.

HOW DO YOU FIND A_n's.

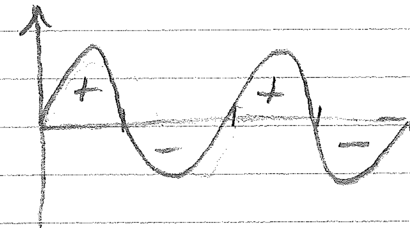
USEFUL FACTS.

$$\int_0^{2\pi} \sin\theta \, d\theta = 0$$

SIN θ



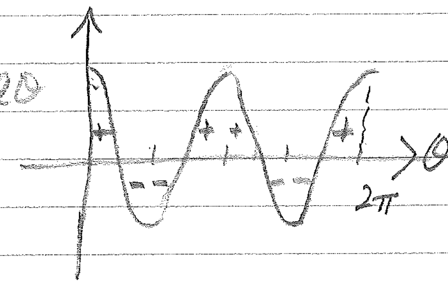
$$\int_0^{2\pi} \sin 2\theta \, d\theta = 0.$$



$$\int_0^{2\pi} \sin^2\theta \, d\theta = \frac{1}{2}\, 2\pi$$

$$\int_0^{2\pi} \sin^2\theta \, d\theta = \int_0^{2\pi} \frac{1-\cos 2\theta}{2}\, d\theta \quad \cos 2\theta \to 0$$

cos 2θ

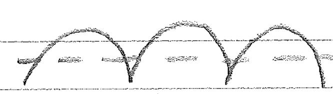


$$\sin ax \sin bx = \frac{1}{2}\left(\cos(a-b)x - \cos(a+b)x\right)$$

$$\int_0^L dx \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L} = \frac{1}{2}\int dx \left\{ \cos\frac{(n-m)\pi x}{L} - \cos\frac{(n+m)\pi x}{L} \right.$$

$= 0$ UNLESS $n = m$.

FOR $n = m$ $\quad \displaystyle\int_0^L \sin^2\frac{n\pi x}{L}\, dx = \frac{1}{2} L.$



$\frac{1}{2} = \langle \sin^2 \rangle$

$\sin^2\frac{n\pi x}{L}$

$$y(x,0) = \sum_m A_m \sin\frac{m\pi x}{L}$$

MULTIPLY BOTH SIDES by $\sin\frac{n\pi x}{L}$ &

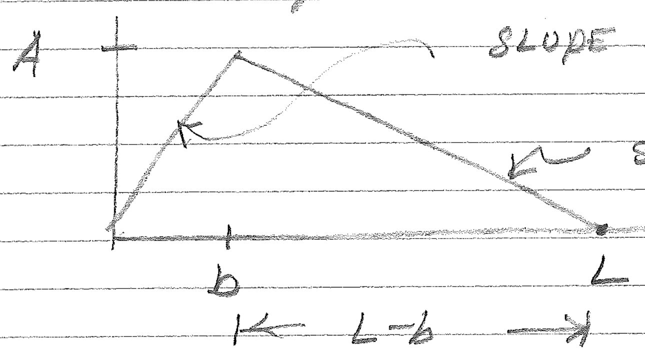
INTEGRATE FROM 0 TO L

$$\int_0^L y(x,0) \sin\frac{n\pi x}{L} = \sum_m \int_0^L A_m \sin\frac{m\pi x}{L} \sin\frac{n\pi x}{L}\, dx$$

$$\boxed{\begin{aligned} &y(x,0) = \sum A_n \sin\frac{n\pi x}{L} \\ &A_n\left(\tfrac{1}{2}L\right) = \int_0^L y(x,0)\,\sin\frac{n\pi x}{L}\,dx \\ &A_n = \frac{2}{L}\int_0^L y(x,0)\,\sin\frac{n\pi x}{L}\,dx \end{aligned}}$$

THIS HAS TO BE GIVEN.

EXAMPLE 1.



SLOPE $\frac{A}{b}$ $\quad y = \frac{A}{b}x$

SLOPE $-\frac{A}{L-b}$ $\quad y = A\frac{L-x}{L-b}$

$x = L \quad y = 0.$

$$A_n\frac{L}{2} = \int_0^b \frac{Ax}{b}\sin\frac{n\pi x}{L}\,dx + \int_b^L A \sin\frac{L-x}{L-b}\,\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{AL^3 \sin\frac{n\pi b}{L}}{(L-b)\,b\,n^2\pi^2} = \frac{AL^3 \sin\frac{n\pi b}{L}}{L^2\left(1-\frac{b}{L}\right)\frac{b}{L}\,n^2\pi^2}$$

$$\boxed{A_n = \frac{2A\,\sin\frac{n\pi b}{L}}{\left(1-\frac{b}{L}\right)\frac{b}{L}}\,\frac{1}{n^2\pi^2}}$$

SPECIFIC CASE $b = L/2$.

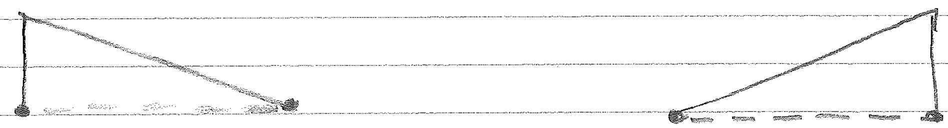
$$\sin\left(\frac{n\pi L/2}{L}\right) = \sin\frac{n\pi}{2} = \begin{cases} 1 & n=1 \quad (\sin\pi/2) \\ 0 & n=2 \quad (\sin\pi) \\ -1 & n=3 \quad (\sin 3\pi/2) \\ 0 & n=4 \quad (\sin 2\pi) \\ 1 & n=5 \\ 0 & \vdots \\ -1 & \vdots \\ 0 & \end{cases}$$

$A_1 = \frac{8A}{\pi^2} \qquad A_3 = -A_1\frac{1}{3^2}$

$A_5 = +A_1\frac{1}{5^2} \quad \ldots$

$A_{2n-1}/A_1 = \frac{1}{(2n-1)^2} \qquad n = 1, 2, 3, \ldots$

WHAT HAPPENS WHEN $b \to 0$ or $b \to L$



$$A_n = 2A \frac{\sin n\pi b/L}{(1-b/L)(b/L)} \frac{1}{n^2\pi^2}$$

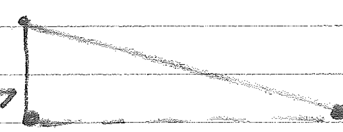
$$b \to 0 \quad \underbrace{\sin n\pi b/L}_{\text{THIS IS SMALL}} \Rightarrow n\pi b/L \qquad (\sin x \simeq x,\ x \ll 1)$$

$$A_n = 2A \frac{n\pi b/L}{(1-0)\, b/L} \frac{1}{n^2\pi^2} = \boxed{2A \frac{1}{n\pi} = A_n}$$

NOW $A_n \sim \frac{1}{n}$ AND NOT $\frac{1}{n^2}$

NEED MANY MORE TERMS IN $\sum A_n \sin \frac{n\pi x}{L}$

TO REPRESENT.



BECAUSE OF
THE VERTICAL
RISE ON LEFT
NEED HIGH f OR SMALL λ

EVOLUTION IN TIME OF THE INITIAL STRING CONFIGURATION

$$y(x,t) = \sum A_n \sin \frac{n\pi x}{L} \cos n\omega t \qquad \omega = 2\pi f = 2\pi \frac{1}{T}$$

$$= \sum A_n \sin \frac{n\pi x}{L} \cos n \frac{2\pi t}{T}$$

$$\boxed{y(x,t) = \sum A_n \sin \frac{2\pi n x}{2L} \cos \frac{2\pi n t}{T} \qquad \lambda_1 = 2L}$$