Quantum field theory and the Ricci flow

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Aside

Two topics I'm interested in at the moment:

- The bundle $G_2 \to S^6 = G_2/SU(3)$ and the Yang-Mills flow.
- Using 1+1d quantum field theories for quantum computation.

Quantum mechanics of a single degree of freedom

Start with one degree of freedom: $q \in \mathbb{R}$.

The Hilbert space of states is $\mathcal{H} = L_2(\mathbb{R})$, consisting of wave functions $\psi(q)$.

The time evolution is generated by a hamiltonian operator H:

$$q(t) = e^{itH} q \, e^{-itH} \qquad H^\dagger = H, \quad H \geq 0$$

The Heisenberg algebra: $\left[i \frac{\partial}{\partial q}, \; q \right] = i$

The free particle hamiltonian:

$$H = \frac{1}{2} \left(-\frac{\partial^2}{\partial q^2} \right)$$

The harmonic oscillator

The harmonic oscillator hamiltonian:

$$H = \frac{1}{2}\left(-\frac{\partial^2}{\partial q^2}\right) + \frac{1}{2}E^2 q^2 \qquad E > 0$$

destruction/creation operators:

$$\mathbf{a} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial q} + Eq\right) \qquad \mathbf{a}^\dagger = \frac{1}{\sqrt{2}}\left(-\frac{\partial}{\partial q} + Eq\right)$$

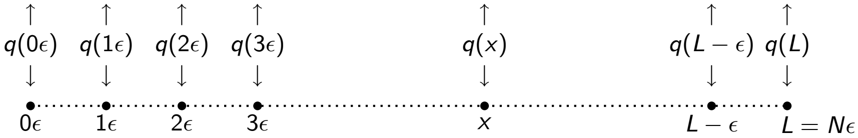
$$[\mathbf{a},\, \mathbf{a}^\dagger] = E$$

$$H = \mathbf{a}^\dagger \mathbf{a} + \frac{1}{2}E \qquad [H,\, \mathbf{a}^\dagger] = E\mathbf{a}^\dagger \qquad [H,\, \mathbf{a}] = -E\mathbf{a}$$

eigenstates:

$$H\psi_n = (nE + \frac{1}{2}E)\psi_n \qquad \mathbf{a}\psi_0 = 0, \quad \psi_n = (\mathbf{a}^\dagger)^n \psi_0, \quad n = 1, 2, \cdots$$

N degrees of freedom in a line



boundary condition: $q(0) = q(L) \quad (\text{i.e., } x \in \epsilon\mathbb{Z}_N)$

Hilbert space: $\mathcal{H} = \bigotimes_x L_2(\mathbb{R}) = L_2(\mathbb{R}^N)$ states: $\psi(q(1\epsilon), q(2\epsilon), \cdots, q(N\epsilon))$

hamiltonian: $$H = \frac{1}{\epsilon}\sum_x \left(-\frac{1}{2}\frac{\partial^2}{\partial q(x)^2} + \frac{1}{2}\left[q(x) - q(x-1)\right]^2\right)$$ **(local !!!)**

$\mathbb{Z}_N$ translation symmetry: $x \mapsto x + \epsilon m \pmod{\epsilon N}, \quad q(x) \mapsto q(x + \epsilon m)$

Equivalent to N harmonic oscillators

Fourier transform in $\mathbb{Z}_N$:

$$\tilde{q}_k = \frac{1}{\sqrt{N}} \sum_x e^{ikx} q(x) \qquad \tilde{p}_k = \frac{1}{\sqrt{N}} \sum_x e^{-ikx} i \frac{\partial}{\partial q(x)} \qquad k = \frac{2\pi m}{\epsilon N} = \frac{2\pi m}{L} \in \frac{1}{\epsilon} \mathbb{Z}_N^*$$

$$[p_k,\ q_{k'}] = i\delta_{k,k'}$$

$$H = \sum_k \frac{1}{2} \tilde{p}_k^\dagger \tilde{p}_k + \frac{1}{2} E_k^2\, \tilde{q}_k^\dagger \tilde{q}_k \qquad E_k = \frac{1}{\epsilon} (2 - 2\cos \epsilon k)^{1/2}$$

$$\mathbf{a}_k = \frac{1}{\sqrt{2}} (-ip_k + E_k q_k) \qquad [\mathbf{a}_k,\ \mathbf{a}_{k'}] = 0 \qquad [\mathbf{a}_k{}^\dagger,\ \mathbf{a}_{k'}] = E_k \delta_{k,k'}$$

$$H = \sum_k \left(\mathbf{a}_k{}^\dagger \mathbf{a}_k + \frac{1}{2} E_k \right) \qquad [H,\ \mathbf{a}_k{}^\dagger] = E_k \mathbf{a}_k{}^\dagger \qquad [H,\ \mathbf{a}_k] = -E_k \mathbf{a}_k$$

The ground state: $\quad \mathbf{a}_k \psi_0 = 0 \qquad H\psi_0 = \sum_k \frac{1}{2} E_k\ \psi_0$

Note that $E_0 = 0$. The zero-mode $\tilde{q}_0$ is a "free particle".

The continuum limit

Send $N \to \infty$, $\epsilon \to 0$ holding $L = N\epsilon$ fixed. Now x parametrizes the circle of length L.

The oscillators are indexed by

$$k = \frac{2\pi m}{\epsilon N} = \frac{2\pi m}{L} \qquad m \in \left\{ -\frac{1}{2}N, \cdots, \frac{1}{2}N \right\}$$

The oscillator energies become

$$E_k = \frac{1}{\epsilon}(2 - 2\cos \epsilon k)^{1/2} \quad \to |k| \qquad \text{for } |k| \ll \frac{1}{\epsilon}$$

The hamiltonian has a limit, once the ground state energy is subtracted,

$$H - \sum_k \frac{1}{2} E_k \quad \to \quad H_{ren} = \sum_{k \in \frac{2\pi}{L}\mathbb{Z}} \mathbf{a}_k{}^\dagger \mathbf{a}_k$$

In taking the limit, we have discarded the states with energy eigenvalues of order $1/\epsilon$.

Quantum fields

Define $\dfrac{\delta}{\delta q(x)} = \dfrac{1}{\epsilon}\dfrac{\partial}{\partial q(x)}$ $\qquad \Delta_x q = \dfrac{q(x) - q(x-\epsilon)}{\epsilon}$

$$H = \epsilon \sum_x \left[-\frac{1}{2}\left(\frac{\delta}{\delta q(x)}\right)^2 + \frac{1}{2}(\Delta_x q)^2 \right]$$

In the continuum limit,

$$\left[\frac{\delta}{\delta q(x)},\ q(x')\right] \quad \to \quad \delta(x - x')$$

$$\Delta_x q \quad \to \quad \frac{\partial q}{\partial x}$$

$q(x)$ and $\dfrac{\delta}{\delta q(x)}$ become operator-valued distributions on the circle, and

$$H - \sum_k \frac{1}{2}E_k \quad \to \quad H_{ren} = \int_0^L dx \left[-\frac{1}{2} : \left(\frac{\delta}{\delta q(x)}\right)^2 : + \frac{1}{2} : \left(\frac{\partial q}{\partial x}\right)^2 : \right]$$

Discussion

- The continuum limit is physically motivated. We quite often investigate physical systems in which many degrees of freedom are distributed locally in space, separated by distances of order ϵ, much smaller than our experimental apparatus can probe.

  With our limited resources, we cannot produce or detect the high energy states that we discarded in taking the continuum limit.

  Everything our instruments can detect will be described to good accuracy by a continuum limit.

- In the continuum limit, what we observe is described by an algebra of operator-valued distributions $\mathcal{O}_\alpha(x)$ smeared by smooth functions. A quantum mechanics with such an algebra of observables (and satisfing a few additional conditions) is a *quantum field theory*.

  Can we say anything about what quantum field theories are possible?
  The only substantial progress on this question has been for qfts in 1+1 dimensions (1 space dimension and 1 time).

Discussion (cont.)

- So far, we have discussed the most trivial example, the free massless scalar field in 1+1 dimensions.

  It is exactly solvable, H being quadratic in Heisenberg operators, so we can take the continuum limit explicitly.

- We have effective techniques – perturbation theory – for studying hamiltonians that are close to free (i.e., close to quadratic).

First, a trivial generalization: $\quad q^i(x) \quad i = 1, \cdots, n$

$$
\begin{aligned}
H &= \frac{1}{\epsilon} \sum_x \sum_{i=1}^{n} \left(-\frac{1}{2} \frac{\partial^2}{\partial q^i(x)^2} + \frac{1}{2} \left[q^i(x) - q^i(x-1) \right]^2 \right) \\
&= \frac{1}{\epsilon} \sum_x \left(-\frac{1}{2} \delta^{ij} \frac{\partial}{\partial q^i(x)} \frac{\partial}{\partial q^j(x)} + \frac{1}{2} \delta_{ij} \Delta_x q^i \Delta_x q^j \right)
\end{aligned}
$$

H is the sum of n free massless scalar hamiltonians. The Hilbert space is the tensor product of n free massless scalar Hilbert spaces.

The path integral

The free particle, $H = \frac{1}{2}\left(-\frac{\partial^2}{\partial q^2}\right)$, as path integral

$$\langle q_T | e^{-TH} | q_0 \rangle = \int \mathcal{D}q \; e^{-S(q)} \qquad S(q) = \int_0^T d\tau \; \frac{1}{2} (\partial_\tau q)^2$$

over paths $q(\tau)$ from $q(0) = q_0$ to $q(T) = q_T$.

- This is just the path integral for the heat kernel.
- Analytically continue in T to get the real time evolution operator e^{-itH}.

Derivation:

$$\langle q_T | e^{-TH} | q_0 \rangle = \int dq_{T-\delta} \cdots \int dq_{2\delta} \int dq_\delta$$

$$\langle q_T | e^{-\delta H} | q_{T-\delta} \rangle \cdots \langle q_{3\delta} | e^{-\delta H} | q_{2\delta} \rangle \langle q_{2\delta} | e^{-\delta H} | q_\delta \rangle \langle q_\delta | e^{-\delta H} | q_0 \rangle$$

$$\langle q_{\tau+\delta} | e^{-\delta H} | q_\tau \rangle = \exp\left[-\frac{\delta}{2} \left(\frac{q_{\tau+\delta} - q_\tau}{\delta} \right)^2 \right]$$

The path integral (2)

Path integral for the harmonic oscillator, $H = \frac{1}{2}\left(-\frac{\partial^2}{\partial q^2}\right) + \frac{1}{2}E^2 q^2$,

$$\langle q_T | e^{-TH} | q_0 \rangle = \int \mathcal{D}q \; e^{-S(q)} \qquad S(q) = \int_0^T d\tau \; \left(\frac{1}{2}(\partial_\tau q)^2 + \frac{1}{2}E^2 q(\tau)^2 \right)$$

For the free massless scalar field,

$$\langle q_T | e^{-TH} | q_0 \rangle = \int \mathcal{D}q \; e^{-S(q)} \qquad S(q) = \int_0^T d\tau \; \epsilon \sum_x \left(\frac{1}{2}(\partial_\tau q)^2 + \frac{1}{2}(\Delta_x q)^2 \right)$$

Now the integral is over paths $\tau \mapsto q(x,\tau)$ for each x, i.e., an integral over all maps $(x,\tau) \mapsto q(x,\tau)$.

Write this in the continuum

$$S(q) = \int_0^T d\tau \int_0^L dx \; \left(\frac{1}{2}\partial_\tau q \partial_\tau q + \frac{1}{2}\partial_x q \partial_x q \right)$$

(leaving the cutoff ϵ implicit).

The path integral (3)

or, writing $x^\mu = (x, \tau)$,

$$S(q) = \int d^2x \; \frac{1}{2} \gamma^{\mu\nu} \partial_\mu q \partial_\nu q$$

where $\gamma_{\mu\nu}$ is the 2d metric:

$$(ds)^2 = \gamma_{\mu\nu} dx^\mu dx^\nu = (dx)^2 + (d\tau)^2$$

A property that was hidden in the hamiltonian formalism becomes manifest:

- 2d euclidean symmetry
- $1 + 1d$ Poincaré symmetry after analytic continuation to real time
- more generally, 2d covariance: $\gamma_{\mu\nu}(x, \tau)$
- *relativistic* quantum field theory

Perturb n free massless scalars

For n free massless scalar fields $q^i(x,\tau)$,

$$S(q) = \int d^2x \, \frac{1}{2}\delta_{ij}\,\gamma^{\mu\nu}\,\partial_\mu q^i\,\partial_\nu q^j$$

We want to investigate the possible continuum limits of small, non-quadratic, perturbations $S \to S + \Delta S$. The perturbations should be local in the $q^i(x,\tau)$ and covariant in 2d.

$$\Delta S = \int d^2x \; q_{k_1} \cdots (\partial_{\mu_1} \cdots q^{i_1}) \cdots$$

Further limit perturbations by requiring them to be *dimensionless* (to be justified later).

Dimensional analysis:

- $\frac{1}{\epsilon}x^\mu$ is a number, so $\dim(x^\mu) = -1$
- $\epsilon\partial_\mu$ is a number, so $\dim(\partial_\mu) = +1$
- $S(q)$ is a number, so $\dim(S) = 0$
- therefore $\dim(q^i) = 0$

So ΔS can contain any number of q^i, and exactly two derivatives ∂_μ.

The general dimensionless perturbation

$$S(q) = \int d^2x \, \frac{1}{2} \left(\delta_{ij} + g_{ij,k} q^k + \frac{1}{2!} g_{ij,k_1 k_2} q^{k_1} q^{k_2} + \cdots \right) \gamma^{\mu\nu} \, \partial_\mu q^i \, \partial_\nu q^j$$

The numbers $g_{ij,k_1 k_2 \cdots}$ are called *coupling constants*.

Define the q-dependent matrix

$$g_{ij}(q) = \delta_{ij} + g_{ij,k} q^k + \frac{1}{2!} g_{ij,k_1 k_2} q^{k_1} q^{k_2} + \cdots$$

so

$$S(q) = \int d^2x \, \frac{1}{2} g_{ij}(q) \, \gamma^{\mu\nu} \, \partial_\mu q^i \, \partial_\nu q^j$$

This is covariant in $\mathbb{R}^n$. The matrix $g_{ij}(q)$ is a Riemannian metric on $\mathbb{R}^n$

$$(ds)^2 = g_{ij}(q) dq^i dq^j$$

The general nonlinear model

Now it is more or less obvious that we should regard $\mathbb{R}^n$ as a coordinate patch on a general n-dimensional manifold M.

The path integral is an integral over maps $(x,\tau) \mapsto q(x,\tau) \in M$

$$\langle q_T | e^{-TH} | q_0 \rangle = \int \mathcal{D}q \; e^{-S(q)} \qquad S(q) = \int d^2x \, \frac{1}{2} g_{ij}(q) \, \gamma^{\mu\nu} \, \partial_\mu q^i \, \partial_\nu q^j$$

$$\int \mathcal{D}q = \prod_{(x,\tau)} \int_M \mathrm{dvol}(q(x,\tau))$$

$$H = \int dx \left(-\frac{1}{2} g^{ij}(q) \frac{\nabla}{\nabla q^i(x)} \frac{\nabla}{\nabla q^j(x)} + \frac{1}{2} g_{ij} \partial_x q^i \partial_x q^j \right)$$

$$\mathcal{H} = \bigotimes_x L_2(M)$$

Again, everything is understood to be cutoff at distance ϵ.

The question is: can we take the continuum limit $\epsilon \longrightarrow 0$?

The general nonlinear model (2)

Introduce a parameter $\alpha' \approx 0$,

$$\int \mathcal{D}q \; e^{-S(q)} \qquad S(q) = \int d^2x \; \frac{1}{2}\frac{1}{\alpha'} g_{ij}(q)\, \gamma^{\mu\nu}\, \partial_\mu q^i \, \partial_\nu q^j$$

The path integral is dominated by the constant maps $q(x,\tau) = q_0 \in M$.

Write $q(x,\tau) = q_0 + \delta q(x,\tau)$

$$\int \mathcal{D}q \; e^{-S(q)} = \int_M dq_0 \int \mathcal{D}\delta q \; e^{-S(q_0+\delta q)}$$

$$S(q_0+\delta q) = \int d^2x \; \frac{1}{2}\frac{1}{\alpha'} g_{ij}(q_0+\delta q)\, \gamma^{\mu\nu}\, \partial_\mu \delta q^i \, \partial_\nu \delta q^j$$

Scale $\delta q^i \to (\alpha')^{1/2}\delta q^i$ to see that this is a free theory plus small perturbations.

We have effective techniques (Feynman diagrams) for systematically evaluating the path integral as a formal power series in α'.

[Simple example: $M = S^1$ (or $M = T^n$). Each fluctuation integral is free, only the zero-mode integral is nontrivial. This example is an excellent exercise.]

The renormalization group flow

Imagine *increasing* ϵ slightly, $\epsilon \to \epsilon + \Delta\epsilon$. The claim is that we can make a compensating change in the action, $S \to S + \Delta S$, so that everything we can observe remains the same.

Start with our original path integral, with cutoff ϵ and action $S(\phi)$. Imagine integrating out a small fraction $2\Delta\epsilon/\epsilon$ of the integration variables $q^i(x,\tau)$. We are left with an integral over the remaining variables $q^i(x,\tau)$, but with a slightly modified action $S + \Delta S$. Now our integration variables are slighlty less dense on the x, τ surface. Essentially, the cutoff is now $\epsilon + \Delta\epsilon$. But nothing observable in the path integral has changed.

In principle, $S + \Delta S$ contains all possible local interactions. Dimensional analysis tells us that terms with more than two derivatives ∂_μ are suppressed by positive powers of ϵ, so we are justified in ignoring them. In the Feynman diagram expansion – the formal power series in α' – this dimensional analysis is rigorous.

So

$$S + \Delta S = S(q) = \int d^2x \, \frac{1}{2} \left[\frac{1}{\alpha'} g_{ij}(q) + \Delta \frac{1}{\alpha'} g_{ij}(q) \right] \gamma^{\mu\nu} \, \partial_\mu q^i \, \partial_\nu q^j$$

$$\Delta \frac{1}{\alpha'} g_{ij}(q) = \left(\frac{\Delta\epsilon}{\epsilon} \right) \beta_{ij}(q)$$

The renormalization group flow (2)

We have constructed a flow

$$\epsilon \frac{\partial}{\partial \epsilon} \frac{1}{\alpha'} g_{ij}(q) = \beta_{ij}(q)$$

such that, if we increase ϵ and move our coupling constants along the flow, nothing observable changes.

There exists a continuum limit if the flow can be run *backwards*, all the way to $\epsilon = 0$. The only circumstance where we know this can be done is when the flow goes backwards to a fixed point $\beta_{ij} = 0$ ("ultraviolet safety"). (We can always define the continuum limit order by order in α'.)

β_{ij} can be calculated as a formal power series in α', to all orders. It is a polynomial in the derivatives of $g_{ij}(q)$, covariant on M, so each term in the formal power series must be a polynomial in the curvature tensor and its covariant derivatives

$$\epsilon \frac{\partial}{\partial \epsilon} \frac{1}{\alpha'} g_{ij} = \beta_{ij} = R_{ij} + \frac{1}{2} \alpha' R_{iklm} R_j^{kjm} + O({\alpha'}^2)$$

A slightly subtlety: the fixed point equation is actually $\beta_{ij} = \nabla_i v_j + \nabla_j v_i$ since a change of metric that is just a reparametrization of the q^i is just a change of variables in the functional integral.

The Ricci flow

$$\frac{d}{dt} g_{ij} = R_{ij}$$

was used by Perelman (following on Hamilton and Thurston) to prove the Poincaré conjecture in $n = 3$ dimensions.

The renormalization group flow for the general nonlinear model

$$\epsilon \frac{\partial}{\partial \epsilon} \frac{1}{\alpha'} g_{ij} = R_{ij} + \frac{1}{2} \alpha' R_{iklm} R_j^{kjm} + O({\alpha'}^2)$$

looks closely related. But note that it does not go to the Ricci flow when $\alpha' \to 0$. We would need to define

$$t = \frac{1}{\alpha'} \ln \epsilon$$

Then the rg flow becomes

$$\frac{d}{dt} g_{ij} g_{ij} = R_{ij} + \frac{1}{2} \alpha' R_{iklm} R_j^{kjm} + O({\alpha'}^2)$$

which does become the Ricci flow in the limit $\alpha' \to 0$.

Elaborations

- more 2d symmetry (2d supersymmetry)
- 1d boundaries and boundary couplings (boundary conditions)

An analytic version of Riemannian manifold?

geometric objects	$\Longrightarrow$	analytic objects
Riemannian manifolds	$\Longrightarrow$	1+1d quantum field theories

Can any mathematical use be made of this?