Supersymmetric 1+1d boundary field theory

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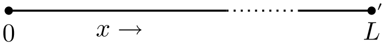
1d quantum systems, critical in the bulk



$$Z = \mathrm{tr}\; e^{-\beta H}$$

for $L/\beta \gg 1$

$$\ln Z = \ln z(\Lambda\beta) + \frac{\pi c}{6}\frac{L}{\beta} + \ln z'$$

boundary renormalization group flow

$$\Lambda \frac{\partial \ln z}{\partial \Lambda} = \beta \frac{\partial \ln z}{\partial \beta} = \beta^a(\lambda)\frac{\partial \ln z}{\partial \lambda^a}$$

Gradient formula for boundary entropy

boundary entropy

$$S = \left(1 - \beta\frac{\partial}{\partial\beta}\right)\ln Z = s + \frac{\pi c}{3}\frac{L}{\beta} + s'$$

gradient formula

$$\frac{\partial s}{\partial\lambda^a} = -g_{ab}\beta^b(\lambda)$$

implying second law of boundary thermodynamics

$$\Lambda\frac{\partial s}{\partial\Lambda} = \beta\frac{\partial s}{\partial\beta} = \beta^a\frac{\partial s}{\partial\lambda^a} = -\beta^a g_{ab}\beta^b \leq 0$$

The boundary behaves as an isolated system.

Supersymmetric 1d systems, critical in the bulk

a conserved fermionic super-charge

$$H = \hat{Q}^2$$

(Advertisement: in cond-mat/0505084 and 0505085, I argued that circuits made of bulk-critical quantum wire, joined at boundaries and junctions, would be ideal for asymptotically large-scale quantum computing: the $c = 24$ monster system in particular.)

A second gradient formula for supersymmetric systems

[DF & A. Konechny, in preparation]

$$\frac{\partial \ln z}{\partial \lambda^a} = -g^S_{ab}\beta^b(\lambda)$$

(the λ^a now restricted to the susy coupling constants)

implying positivity of the susy boundary energy

$$\Lambda \frac{\partial \ln z}{\partial \Lambda} = \beta \frac{\partial \ln z}{\partial \beta} = \beta^a \frac{\partial \ln z}{\partial \lambda^a} = -\beta^a g^S_{ab}\beta^b \leq 0$$

The boundary behaves as an isolated supersymmetric system.

Here, I will prove directly the positivity of the boundary energy

$$\Lambda \frac{\partial \ln z}{\partial \Lambda} = \beta \frac{\partial \ln z}{\partial \beta} \leq 0$$

equivalently, that $\ln z$ decreases under the RG flow.

Local densities

energy and super-charge

$$H = \int_0^L dx\, \mathcal{H}(t,x)$$

$$\hat{Q} = \int_0^L dx\, \hat{\rho}(t,x)$$

$$[\hat{Q},\ \hat{\rho}(t,x)]_+ = 2\mathcal{H}(t,x)$$

local conservation of super-charge

$$\partial_t \hat{\rho}(t,x) + \partial_x \hat{\jmath}(t,x) = 0$$

Boundary energy and super-charge

$$h(t) = \lim_{\epsilon \longrightarrow 0} \int_0^{\epsilon} dx\, \mathcal{H}(t,x)$$

$$\hat{q}(t) = \lim_{\epsilon \longrightarrow 0} \int_0^{\epsilon} dx\, \hat{\rho}(t,x)$$

$$[\hat{Q},\ \hat{q}(t)]_{+} = 2h(t)$$

$$-\frac{\partial \ln z}{\partial \beta} = \langle h \rangle$$

Separate $\hat{Q}$ into boundary and bulk parts at $x = \epsilon$

$$\hat{q}_\epsilon(t) = \int_0^\epsilon dx\ \hat{\rho}(t,x) \qquad \hat{Q}_{bulk}(t) = \int_\epsilon^L dx\ \hat{\rho}(t,x)$$

$$\hat{Q} = \hat{q}_\epsilon(t) + \hat{Q}_{bulk}(t)$$

locality implies

$$[\hat{Q}_{bulk}(0),\ \hat{q}(0)]_+ = 0$$

so

$$\langle 2h \rangle = \langle [Q,\ \hat{q}(0)]_+ \rangle = \langle [\hat{q}_\epsilon(0),\ \hat{q}(0)]_+ \rangle$$

but this equation is useless at $\epsilon = 0$, because

$\langle [\hat{q}(t),\ \hat{q}(0)]_+ \rangle$ is uv divergent at $t = 0$.

The boundary cannot be separated from the bulk, in general.

Use bulk super-conformal invariance

define

$$g_\epsilon(\omega) = \int_{-\infty}^{\infty} dt\; e^{i\omega t} \left\langle [\hat{q}_\epsilon(t),\, \hat{q}(0)]_+ \right\rangle$$

$$G_\epsilon^\pm(\omega) = \pm \int_0^{\pm\infty} dt\; e^{i\omega t} \left\langle [\hat{Q}_{bulk}(t),\, \hat{q}(0)]_+ \right\rangle$$

so

$$2\pi\delta(\omega)\left\langle 2h \right\rangle = g_\epsilon(\omega) + G_\epsilon^+(\omega) + G_\epsilon^-(\omega)$$

bulk super-conformal invariance implies

$$G_\epsilon^+(i\pi/\beta) = 0 = G_\epsilon^-(-i\pi/\beta)$$

so

$$\int \frac{d\omega}{2\pi}\; \frac{\pi^2/\beta^2}{\omega^2 + \pi^2/\beta^2}\; G_\epsilon^\pm(\omega) = 0$$

so

$$\langle 2h \rangle = \int \frac{d\omega}{2\pi} \, \frac{\pi^2/\beta^2}{\omega^2 + \pi^2/\beta^2} \, g_\epsilon(\omega)$$

Now take $\epsilon \to 0$:

$$g(\omega) = \int_{-\infty}^{\infty} dt \; e^{i\omega t} \left\langle [\hat{q}(t), \, \hat{q}(0)]_+ \right\rangle$$

$$\beta \frac{\partial \ln z}{\partial \beta} = -\beta \langle h \rangle = -\frac{\beta}{2} \int \frac{d\omega}{2\pi} \, \frac{\pi^2/\beta^2}{\omega^2 + \pi^2/\beta^2} \, g(\omega)$$

which is uv-finite as long as $\dim[g(\omega)] < 1$, i.e., $\dim[\hat{q}] < 1$.

We have $g(\omega) \geq 0$ and $g(\omega) = 0$ iff $\hat{q} = 0$, so

$$\beta \frac{\partial \ln z}{\partial \beta} \leq 0$$

with equality iff the boundary is critical (superconformal).

The gradient formula

boundary operators

$$[\hat{Q}, \hat{\phi}_a(t)]_+ = \phi_a(t)$$

$$\frac{\partial \ln z}{\partial \lambda^a} = \beta \langle \phi_a \rangle$$

boundary beta-functions

$$\hat{q} = -2\beta^a \hat{\phi}_a$$

$$h = \frac{1}{2}[\hat{Q}, \hat{q}]_+ = -\beta^a \phi_a$$

$$\Lambda \frac{\partial \ln z}{\partial \Lambda} = \beta \frac{\partial \ln z}{\partial \beta} = -\beta \langle h \rangle = \beta \langle \beta^a \phi_a \rangle = \beta^a \frac{\partial \ln z}{\partial \lambda^a}$$

$$\langle \phi_a \rangle = \langle [\hat{Q}, \hat{\phi}_a(0)]_+ \rangle = \langle [\hat{q}_\epsilon(t) + \hat{Q}_{bulk}(t), \hat{\phi}_a(0)]_+ \rangle$$

$$\begin{aligned} g_a(\omega) &= \int_{-\infty}^{\infty} dt\ e^{i\omega t} \langle [\hat{q}(t), \hat{\phi}_a(0)]_+ \rangle \\ &= \int_{-\infty}^{\infty} dt\ e^{i\omega t} \langle [-2\beta^b \hat{\phi}_b(t), \hat{\phi}_a(0)]_+ \rangle \\ &= -2\beta^b g_{ab}(\omega) \end{aligned}$$

$$\begin{aligned} \langle \phi_a \rangle &= \int \frac{d\omega}{2\pi} \frac{\pi^2/\beta^2}{\omega^2 + \pi^2/\beta^2}\, g_a(\omega) \\ &= -2\beta^b \int \frac{d\omega}{2\pi} \frac{\pi^2/\beta^2}{\omega^2 + \pi^2/\beta^2}\, g_{ab}(\omega) \end{aligned}$$

$$\frac{\partial \ln z}{\partial \lambda^a} = -g^S_{ab}\beta^b$$

$$g_{ab}(\omega) = \int_{-\infty}^{\infty} dt\; e^{i\omega t}\left\langle [\hat{\phi}_b(t),\, \hat{\phi}_a(0)]_+ \right\rangle$$

$$\begin{aligned}
g^S_{ab} &= 2\beta \int \frac{d\omega}{2\pi}\; \frac{\pi^2/\beta^2}{\omega^2 + \pi^2/\beta^2}\; g_{ab}(\omega) \\
&= \pi \int dt\; e^{-\pi|t|/\beta}\left\langle [\hat{\phi}_b(t),\, \hat{\phi}_a(0)]_+ \right\rangle \\
&= 2\pi \int_0^{\beta} d\tau\; \sin\left(\frac{\pi\tau}{\beta}\right)\left\langle \hat{\phi}_b(-i\tau),\, \hat{\phi}_a(0) \right\rangle
\end{aligned}$$

Some questions

1. Why do we need bulk conformal invariance?

2. Why do we need canonical uv behavior in the boundary?
   - no negative dimension boundary operators
   - no strongly irrelevant boundary operators

3. Does the result apply to composite boundaries/junctions?

4. Can $\ln z$ (and/or s) be bounded below?

Bulk conformal invariance and zeros of response functions

$$\partial_t \hat{Q}_{bulk}(t) = \int_\epsilon^L dx \, [-\partial_x \hat{\jmath}(t,x)] = \hat{\jmath}(t,\epsilon)$$

Define response functions

$$R_a^\pm(\omega) = \pm \int_0^{\pm\infty} dt \; e^{i\omega t - \delta|t|} \langle [i\hat{\jmath}(t,\epsilon), \, \hat{\phi}_a(0)]_+ \rangle$$

$R_a^+(\omega)$ is analytic in the upper half-plane, $R_a^-(\omega)$ in the lower.

Use the conservation equation

$$G_{a,\epsilon}^\pm(\omega) = \pm \int_0^{\pm\infty} dt \; e^{i\omega t - \delta|t|} \langle [\hat{Q}_{bulk}(t), \, \hat{\phi}_a(0)]_+ \rangle = \frac{R_a^\pm(\omega)}{\omega \pm i\delta}$$

$\tau = it$, $0 < \tau < \beta$

$$\left\langle \hat{\jmath}(-i\tau, \epsilon)\, \hat{\phi}_a(0) \right\rangle = \int \frac{d\omega}{2\pi i}\, \frac{e^{-\omega\tau}}{1 + e^{-\omega\beta}} \left[R^+(\omega) + R^-(\omega) \right]$$

poles at

$$\omega_n = \frac{2\pi i n}{\beta} \qquad n \in \frac{1}{2} + \mathbb{Z}$$

so

$$\left\langle \hat{\jmath}(-i\tau, \epsilon)\, \hat{\phi}_a(0) \right\rangle = \beta^{-1} \sum_n e^{-\omega_n \tau} \left[\theta(n) R^+(\omega_n) - \theta(-n) R^-(\omega_n) \right]$$

but

$$j(-i\tau, x) = AG(x + i\tau) + \bar{A}G(x - i\tau)$$

so

$$R_a^+(i\pi/\beta) = 0 = R_a^-(-i\pi/\beta)$$