The Space of Conformal Boundary Conditions for the $c = 1$ Gaussian Model (more)

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The space B of conformal boundary conditions for the $c = 1$ gaussian model is a fiber space $B \to B_0$.

B_0 is a quotient $SU(2)/\mathrm{Z}\times\mathrm{Z}$ of $SU(2)$ by the following action of $\mathrm{Z}\times\mathrm{Z}$.

Let R be the radius of the target circle of the gaussian model. Duality is $R \to 1/R$. The self-dual model at $R = 1$ is the $SU(2)$ level 1 conformal field theory.

Parametrize the elements $g \in SU(2)$ by a pair of complex numbers (u, v) with $|u|^2 + |v|^2 = 1$,

$$g = \begin{pmatrix} u & v \\ -\bar{v} & u \end{pmatrix} \tag{1}$$

$(n_1, n_2) \in \mathrm{Z}\times\mathrm{Z}$ acts on $SU(2)$ by

$$(u, v) \mapsto (e^{2\pi i n_1/R} u, e^{2\pi i n_2 R} v) \tag{2}$$

or, equivalently,

$$g \mapsto U\, g\, V^{-1} \tag{3}$$

where

$$U = \begin{pmatrix} e^{i\pi(n_1/R+n_2R)} & 0 \\ 0 & e^{-i\pi(n_1/R+n_2R)} \end{pmatrix} \tag{4}$$

$$V = \begin{pmatrix} e^{i\pi(-n_1/R+n_2R)} & 0 \\ 0 & e^{-i\pi(-n_1/R+n_2R)} \end{pmatrix} \tag{5}$$

The full space B of conformal boundary conditions is a quotient of a fiber space $\widetilde{SU(2)}$ over $SU(2)$ by a compatible action of $\mathrm{Z}\times\mathrm{Z}$.

To make $\widetilde{SU}(2)$, remove the circle $v=0$ from $SU(2)$ and replace it by its covering space, the real line. Parametrize this real line by θ_1, with covering map $\theta_1 \mapsto (e^{i\theta_1}, 0)$.

Likewise, remove the circle $u=0$ and replace it by its covering space, another real line. Parametrize this real line by θ_2, with covering map $\theta_2 \mapsto (0, e^{i\theta_2})$.

$\widetilde{SU}(2)$ is a covering group of $SU(2)$ in the obvious way. As a point set, $\widetilde{SU}(2) = \mathrm{R} \cup (SU(2) - S^1 - S^1) \cup \mathrm{R}$. Its topology is described below.

Let $(n_1, n_2) \in \mathrm{Z}\times\mathrm{Z}$ act on $\widetilde{SU}(2)$ by

$$(u,v) \mapsto (e^{2\pi i n_1/R}u, e^{2\pi i n_2 R}v) \quad (6)$$

$$\theta_1 \mapsto \theta_1 + 2\pi n_1/R \quad (7)$$

$$\theta_2 \mapsto \theta_2 + 2\pi n_2 R \quad (8)$$

As before, this action has the form $\tilde{g} \mapsto \tilde{U}\tilde{g}\tilde{V}^{-1}$ for $\tilde{g} \in \widetilde{SU}(2)$, where the $\tilde{U}$ and the $\tilde{V}$ represent the gaussian model momentum lattice and dual lattice in $\widetilde{SU}(2)$.

B is the quotient of $\widetilde{SU}(2)$ by this action of $\mathrm{Z}\times\mathrm{Z}$.

The actions of $\mathrm{Z}\times\mathrm{Z}$ are compatible with the covering $\widetilde{SU}(2) \to SU(2)$, so the quotient is a fiber space $B \to B_0$.

The circle $u=0$ in $SU(2)$ becomes in B_0 a quotient S^1/Z where Z acts by $v \to e^{2\pi i n_2 R}v$. The fiber in B is a circle of radius R. These are the Dirichelet boundary conditions.

The circle $v=0$ in $SU(2)$ becomes in B_0 a quotient S^1/Z, where Z acts by $u \to e^{2\pi i n_1/R}u$. The fiber in B is a circle of radius $1/R$. These are the Neumann boundary conditions (the Dirichelet boundary conditions of the dual $R \to 1/R$ model).

The topology of B is not, in general, Hausdorff. Recall that B is constructed as the spectrum of a commutative C^*-algebra. B is the set of extreme points of a compact convex space, which is then interpreted as the space of positive measures of weight 1 on B. B is embedded in the space of probability measures. A converging sequence in B can have a limit which is not a point in B, but a probability measure on B.

The actions of $\mathrm{Z}\times\mathrm{Z}$ are actions in the sense of measure, averaging actions. When R is rational, the actions of $\mathrm{Z}\times\mathrm{Z}$ factor through a finite group. Then the action on point sets is the same as the measure theoretic action. When R is irrational, the actions of $Z\times Z$ are ergodic. For example, the circle $u=0$ in $SU(2)$ is taken to a single point, represented by the measure which averages over the ergodic action of Z. The averaging action becomes a probability integral. Presumably, there is an elegant interpretation of all this in terms of noncommutative geometry.

The fiber space $\widetilde{SU(2)} \to SU(2)$ has a measure theoretic structure, as follows. A sequence in $SU(2) - S^1 - S^1$ that converges to $(0,v)$ in $SU(2)$ converges in $\widetilde{SU(2)}$ to the averaging measure

$$f(\theta_2) \mapsto \mathrm{Av}_{n_2\in \mathrm{Z}}\, f(\theta_2 + 2\pi n_2) \tag{9}$$

Topologically, the sequence converges to the infinite discrete set of points θ_2 in $\widetilde{SU(2)}$ that lie over $(0,v)$. The limit of the converging sequence is an average over Z acting on the real line. This average is perhaps a bit delicate to define properly. But there is no difficulty in B, where the real line has become the circle of radius R (θ_2 mod $2\pi R$) and the average is over θ_2 mod 2π.

Annulus calculations

The conformal boundary conditions $b\in B$ correspond to normalized boundary states $\langle b|$, normalized so that the partition function of the disk is $\langle b|0\rangle = 1$.

The partition function of the annulus with normalized boundary conditions b_1, $b_2\in B$

$$F(b_1,b_2) = \langle b_1|e^{i\pi\tau(L_0+\bar{L}_0)}|b_2\rangle \tag{10}$$

is calculated by averaging over Z$\times$Z.

Write B_G for the subset $S^1\cup S^1$ in B, the Dirichelet and Neumann (dual Dirichelet) boundary conditions, the circles of radius R and $1/R$.

If b_1 and b_2 are both in B_G, then the annulus partition function $F(b_1,b_2)$ is the usual partition function of the gaussian model with normalized Dirichelet and Neumann boundary conditions.

If either b_1 or b_2 is in $B-B_G$ then $F(b_1,b_2)$ depends only on the projections $\pi(b_1)$, $\pi(b_2)$ of b_1, b_2 to boundary states in $B_0 = SU(2)/\mathrm{Z}\times\mathrm{Z}$. The annulus partition function is given by averaging the $SU(2)$ annulus partition functions ($R=1$) over the action of Z×Z

$$
\begin{aligned}
F(b_1,b_2) &= \mathrm{Av}_{U,V\in \mathrm{Z}\times\mathrm{Z}}\ F(g_1, U g_2 V^{-1})_{SU(2)} && (11)\\
&= \mathrm{Av}_{U,V}\ \frac{1}{\eta(\tau)} \sum_j \left(e^{2\pi i\tau j^2} - e^{2\pi i\tau (j+1)^2}\right)\ \mathrm{tr}\,\rho_j(g_1^{-1} U g_2 V^{-1}) && (12)\\
&= \mathrm{Av}_{U,V}\ \frac{1}{\eta(\tau)} \sum_{n\in \mathrm{Z}} e^{2\pi i\tau n^2/4}\ e^{in\theta(g_1^{-1} U g_2 V^{-1})} && (13)
\end{aligned}
$$

where $g_1 \in SU(2)$ is a representative of $\pi(b_1) \in SU(2)/\mathrm{Z}\times\mathrm{Z}$ and $g_2 \in SU(2)$ is a representative of $\pi(b_2) \in SU(2)/\mathrm{Z}\times\mathrm{Z}$, and where ρ_j is the spin j representation of $SU(2)$ and $\theta(g)$ is given by

$$
2\,\cos\,\theta(g) = \mathrm{tr}(g) \qquad (14)
$$

for $g \in SU(2)$.

The previous case, when both b_1 and b_2 are in B_G, can likewise be written as an average over Z×Z acting on free field boundary conditions (with a bit of delicacy needed to take the average).

Before averaging over Z×Z, the boundary conditions correspond to the elements of $\widetilde{SU}(2)$. I expect that calculations with more than two boundary circles will follow the same pattern: first calculate with an element of $\widetilde{SU}(2)$ on each boundary circle, then average over the action of Z×Z for each boundary circle.

The disk partition function $Z(b)$

The unnormalized boundary state corresponding to $b \in B$ is $Z(B)\,\langle b|$, where the number $Z(b)$ is the partition function of the disk with boundary condition b.

The annulus is the finite temperature partition function of the cft on the interval with boundary condition b on both ends. The limit $\tau \to 0$ is the limit of zero temperature. The infinitesimally thin annulus is conformal to the infinitely long strip, which is conformal to the disk. The two boundary circles of the annulus join to become the single boundary circle of the disk. The condition that the ground state of the cft on the interval occurs with multiplicity 1 in the partition function determines the number $Z(b)$ by the equation :

$$Z(b)^2 \, F(b,b) \to \tilde{q}^{-1/24} \tag{15}$$

as $\tau \to 0$ where $\tilde{q} = e^{-2\pi i/\tau}$.

The Neumann boundary conditions b_N, lying in the circle of radius $1/R$ at $v = 0$, have

$$Z(b_D) = 2^{-1/4} \, R^{1/2} \tag{16}$$

The Dirichelet boundary conditions b_D, lying in the circle of radius R at $u = 0$, have

$$Z(b_N) = 2^{-1/4} \, R^{-1/2} \tag{17}$$

For non-Dirichelet, non-Neumann boundary conditions $b \in B - B_G$, at $u, v \neq 0$, the disk partition function $Z(b)$ is finite only if R is a rational number. If R is rational, with $R = P/Q$, P and Q relatively prime,

$$Z(b) = 2^{-1/4} \, (P\,Q)^{1/2} \tag{18}$$

When R is irrational, the annulus partition function in the limit $\tau \to 0$ shows a dense spectrum at energies near the ground state energy of the strip, so $Z(b)$ is infinite.

When R is irrational, the average over Z×Z becomes a continuous average over two angles, $e^{i\alpha_1} = e^{in_1/R}$ and $e^{i\alpha_2} = e^{in_2 R}$. Only the regions $\alpha_1 \approx 0$ and $\alpha_2 \approx 0$ contribute to the low energy spectrum of the cft on the interval.

When $-1/\tau \to \infty$, change variables in the averaging to

$$k_1 = \frac{1}{2\pi} \, |u| \, \alpha_1 \qquad k_2 = \frac{1}{2\pi} \, |v| \, \alpha_2 \tag{19}$$

to obtain

$$F(b,b) \approx \; 2^{1/2} \, \frac{1}{|uv|} \int dk_1 \int dk_2 \; (\tilde{q})^{k_1^2 + k_2^2 - 1/24} \tag{20}$$

where this description of the low energy spectrum is accurate in the region $|k_1| \ll |u|$, $|k_2| \ll |v|$.

Two unforeseen noncompact dimensions appear at low momentum in the open string sector!