CONFORMAL INVARIANCE, UNITARITY AND TWO DIMENSIONAL CRITICAL EXPONENTS[†]

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ABSTRACT

We show that conformal invariance and unitarity severely limit the possible values of critical exponents in two dimensional systems by finding the discrete series of unitarisable representations of the Virasoro algebra. The realization of conformal symmetry in a given system is parametrized by a real number c, the coefficient of the trace anomaly. For $c<1$ the only values allowed by unitarity are $c=1-6/m(m+1)$, $m=2,3,4\cdots$. For each of these values of c unitarity determines a finite set of rational numbers that must contain all possible critical exponents. These finite sets account for the known critical exponents of the following two dimensional models: Ising(m=3), tricritical Ising(m=4), 3-state Potts(m=5), and tricritical 3-state Potts(m=6).

1. INTRODUCTION

One of the most intriguing features of statistical mechanical systems and of their analogs, euclidean field theories, is the existence of special critical points where the systems are scale invariant. Correlation functions of the fluctuating fields mirror this lack of scale by transforming very simply under dilations of space. The two point function $\langle\phi(\vec{r})\phi(0)\rangle$, for example, is proportional to r^{-2x}. The number x is called the scaling dimension of ϕ (the anomalous dimension in field theory). The scaling dimensions of the fields determine the critical exponents of the system.

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The renormalization group has provided a satisfying conceptual framework for understanding the occurrence of scale invariant points. They are fixed points of the system under renormalization group transformations. The scaling dimensions describe the linearized behavior of the renormalization group near the fixed point. Although this viewpoint is elegant and compelling, it provides no general procedure for classifying or constraining possible fixed points. Calculations are done within the context of specific families of models.

Many known examples of fixed points display a richer symmetry than simple scale invariance. They are invariant under local rescalings: transformations of space that preserve angles but change lengths differently at different points. These are the conformal transformations. The question arises whether conformal invariance can be used to constrain or construct possible fixed point theories. This possibility was suggested by Polyakov[1].

Two dimensions is an especially promising place to apply notions of conformal invariance, because there the group of conformal transformations is infinite dimensional. Any analytic function mapping the complex plane to itself is conformal. Belavin, Polyakov and Zamolodchikov (BPZ)[2] have shown how the rich structure of the conformal group in two dimensions can be used to analyze conformally invariant field theories.

Many two dimensional statistical mechanical systems can also be interpreted as 1+1 dimensional quantum field theories. The distinguishing feature of the quantum theories is unitarity, equivalent to the property of reflection positivity in the statistical systems. We shall see that the quantum mechanical condition of unitarity, in the presence of the large conformal transformation group, puts a powerful constraint on the allowed physical systems.

2. CONFORMAL INVARIANCE IN TWO DIMENSIONS

Conformal invariance is simplest to describe in complex coordinates $z=x+iy$, $\bar{z}=x-iy$. The infinitesimal conformal transformations are $z \longrightarrow z+v(z)$, $\bar{z} \longrightarrow \bar{z}+\bar{v}(\bar{z})$. A simple basis is $v(z)=\epsilon z^{n+1}$, $\bar{v}(\bar{z})=\bar{\epsilon}\bar{z}^{n+1}$. When $n=-1,0,1$ these generate the group of fractional linear transformations $z \longrightarrow w(z)=(az+b)/(cz+d)$, which are the only globally defined conformal transformations on the Riemann sphere. This group is abbreviated SL_2.

Conformally invariant systems are described by correlation functions of a collection of conformal fields. Conformal fields are tensors of the form $\phi(z,\bar{z})dz^h d\bar{z}^{\bar{h}}$, which transform infinitesimally as

$$(1) \qquad \phi \longrightarrow \phi + \left[v(z)\frac{\partial}{\partial z} + hv'(z) + \bar{v}(\bar{z})\frac{\partial}{\partial \bar{z}} + \bar{h}\bar{v}'(\bar{z})\right]\phi.$$

The correlation functions are invariant under the SL_2 transformations. The two point function, for example, obeys

$$(2) \qquad \langle\phi(z_1)\phi(z_2)\rangle = w'(z_1)^h \bar{w}'(\bar{z}_1)^{\bar{h}} w'(z_2)^h \bar{w}'(\bar{z}_2)^{\bar{h}} \langle\phi(w(z_1))\phi(w(z_2))\rangle.$$

From the invariance under Euclidean and scale transformations we get $\langle\phi(re^{i\theta})\phi(0)\rangle = r^{-2(h+\bar{h})}e^{-2i\theta(h-\bar{h})}$, so $h+\bar{h}$ is x, the scaling dimension of ϕ, and $h-\bar{h}$ is its spin.

The consequences of conformal variations with n not equal $-1,0,1$ are summarized by saying that there is a field $T_{\mu\nu}(x,y)$, called the stress energy tensor, which generates local scale transformations. Translation invariance and global scale invariance require $T_{\mu\nu}$ to be conserved and traceless; in complex notation we can write $T_{\mu\nu}dx^\mu dx^\nu = T(z)dz^2 + \bar{T}(\bar{z})d\bar{z}^2$. The requirement that T generate local scale transformations is expressed in the operator product of T with conformal fields:

$$(3) \qquad T(w)\phi(z,\bar{z}) \sim h(w-z)^{-2}\phi(z,\bar{z}) + (w-z)^{-1}\partial_z\phi(z,\bar{z})$$

and similarly for $\bar{T}$. The action of conformal transformations on correlation functions can then be expressed as

$$(4) \qquad \frac{1}{2\pi i}\oint_C dw\; v(w)T(w)\;\phi(z,\bar{z}) = \left[v(z)\frac{\partial}{\partial z} + hv'(z)\right]\phi(z,\bar{z})$$

where the contour surrounds z. A similar equation holds for $\bar{T}$. The splitting of the z and $\bar{z}$ dependence is a noteworthy simplification.

The action of conformal transformations on T itself is given by the operator product:

$$(5) \qquad T(w)T(z) \sim \tfrac{1}{2}c(w-z)^{-4} + 2(w-z)^{-2}T(w) + (w-z)^{-1}T'(w)$$

The form of (5) is forced by equation (3) and SL_2 invariance. $\bar{T}$ satisfies the same relation, assuming the theory is $z \longleftrightarrow \bar{z}$ invariant.

The number c measures the deviation of T from transforming as a true conformal field for $n \neq -1,0,1$. It also describes the lack of invariance of the ground state to curving the underlying space - an effect referred to as the trace anomaly[3].

3. HILBERT SPACE INTERPRETATION

There exist general conditions on a system that allow it to be reconstructed as a theory of operator fields acting on a space of states with a hermitian inner product. The correlation functions are interpreted as vacuum expectation values of time-ordered products of operator fields. Some of the most important models have the property that the inner product on the space of states is positive definite, i.e. the state space is a Hilbert space. This unitarity property allows such models to be interpreted as quantum field theories. Models where a hermitian transfer matrix construction can be made provide concrete examples.

The positivity of the metric in the Hilbert space of states is equivalent to the property of reflection positivity in the correlation functions. Reflection positivity is imposed by singling out a hypersurface and requiring that correlations of products of fields on one side of the hypersurface with their reflections on the other side be nonnegative. The hypersurface is typically interpreted as constant in time. In two dimensions the hypersurface would be a line and the reflection would be $z \longleftrightarrow \bar{z}$.

The infinitesimal conformal transformations $\delta z = \epsilon z^{n+1}$ are singular for $n \neq -1,0,1$ either at the origin or at infinity, so it is useful to define operators on an alternate hypersurface, the unit circle, where z^{n+1} is always well-behaved. The reflection operation is

$z \leftrightarrow 1/\bar{z}$. Because correlation functions are SL_2 invariant, and because the unit circle can be mapped by an SL_2 transformation to a line, reflection positivity through a line is equivalent to reflection positivity through the unit circle.

We can express correlation functions for quantum field theories as operator expectation values in this nonstandard Hilbert space. Dilation takes the place of time translation and radial ordering takes the place of time ordering. Alternatively, using coordinates $z=e^{\tau+i\theta}$, this is quantum field theory in Euclidean "time" τ and a periodic one dimensional space θ.

We define the operators that implement conformal transformations by

$$(6) \qquad T(z)=\sum_{n=-\infty}^{\infty} z^{-n-2}L_n \quad \bar{T}(\bar{z})=\sum_{n=-\infty}^{\infty} \bar{z}^{-n-2}\bar{L}_n.$$

It follows from (3), (5) and hermiticity of the inner product that the stress energy tensor is self-adjoint, i.e. $[T(1/\bar{z})d(1/\bar{z})^2]^\dagger=T(z)dz^2$, and the equivalent for $\bar{T}$. This translates to

$$(7) \qquad L_n^\dagger=L_{-n} \quad \bar{L}_n^\dagger=\bar{L}_{-n}.$$

Formula (3) rewritten in operator language reads

$$(8) \qquad [L_n,\phi] = z^{n+1}\frac{\partial}{\partial z}\phi + h(n+1)z^n\phi.$$

Dilations are generated by $L_0+\bar{L}_0$, rotations by $L_0-\bar{L}_0$, and translations by L_{-1} and $\bar{L}_{-1}$. The SL_2 invariance of the correlation functions implies SL_2 invariance of the vacuum: $L_n|0\rangle=0=\bar{L}_n|0\rangle$, for $n=-1,0,1$.

Equation (5) is equivalent to the commutation relations

$$(9) \qquad [L_m,L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n}.$$

The operators $\bar{L}_n$ satisfy the same algebra and commute with all the L's. The algebra (9) is called the Virasoro algebra[4]. It was first encountered in string theory where conformal invariance is a residue of full reparametrization invariance in a certain (conformal) gauge[5].

The central term with coefficient c in (9) describes the particular realization of conformal symmetry in the theory. We see that the product of two Virasoro algebras each with central charge c acts on the space of states of the conformal field theory. Reflection positivity tells us that this is a unitary representation.

If a field ϕ satisfies the transformation law (8), at least for n=0, then $L_0\phi(0)|0\rangle=h\phi(0)|0\rangle$. Thus the problem of finding critical exponents is reduced to understanding the allowed eigenvalues of L_0. To constrain these values we must remember that the states of the Hilbert space are not only eigenstates of L_0; they also form a unitary representation of the Virasoro algebra.

4. REPRESENTATIONS OF THE VIRASORO ALGEBRA

We focus on representations of one Virasoro algebra since representations of the product will just be tensor products of representations of the factors. First, the L_n for $n>0$ are lowering operators for L_0, i.e. $L_0L_n=L_n(L_0-n)$. The vacuum must have the lowest eigenvalue of L_0, so it is annihilated by all the L_n for $n>0$, in addition to the SL_2 generators $n=-1,0,1$. Each conformal operator ϕ can be associated with a state $|h\rangle=\phi(0)|0\rangle$. By (8), this state satisfies $L_0|h\rangle=h|h\rangle$ and $L_n|h\rangle=0$ for $n>0$. A state such as $|h\rangle$ which is annihilated by all the lowering operators is called a highest weight vector. There is a one to one correspondence between the highest weight vectors and the conformal fields of the theory.

Once we have a highest weight vector we can build a representation of the Virasoro algebra by applying the L_{-n}, $n\geq 1$. These states can be classified by L_0 eigenvalue. A state is in the n^{th} level if its L_0 value is h+n. A basis of states at the n^{th} level is given by

$$(10)\qquad \{L_{-k_1}L_{-k_2}\cdots L_{-k_m}|h\rangle:\ \Sigma k_i=n,\ k_1\geq k_2\geq\cdots\geq k_m>0\}.$$

There are P(n) such states, where the classical partition function P(n) is the number of ways of writing n as a sum of positive integers. The tower of such levels is called a highest weight representation of the algebra.

In terms of field theory, these higher level states correspond to operators of higher scaling dimension, obtained by applying products of stress-energy tensors to some conformal field. We should think of each conformal field as carrying such a conformal family along with it. The organization of all the fields of the theory into conformal families, each associated with a conformal field, can be accomplished by making repeated operator products of $T(z)$ with an arbitrary field. The operator coefficient of the most singular term will eventually obey the defining relation (3) of a conformal field.

The inner product of any two states in the span of basis (10) can be computed from the hermiticity condition (7) and the commutation relations (9). The unitarity constraint is that the matrix of inner products should have no negative eigenvalues. We can impose the positivity constraint level by level because different levels have different L_0 eigenvalues and hence are orthogonal. A state $|\psi\rangle$ in the span of basis (10) with $\langle\psi|\psi\rangle$ negative is called a "ghost." If a ghost is found on any level the representation cannot occur in any unitary theory.

At level 1 there is a single state, $|1\rangle = L_{-1}|h\rangle$, and $\langle 1|1\rangle = 2h$. Therefore positivity at level 1 rules out all $h<0$. At level n the state $|n\rangle = L_{-n}|h\rangle$ has $\langle n|n\rangle = 2nh + cn(n^2-1)/12$. If $c<0$ this is a ghost state for large n. So we can limit our attention to the region $c \geqslant 0$, $h \geqslant 0$.

5. THE KAC DETERMINANT AND UNITARITY

To proceed we employ a version of Kac's formula for the determinant of the n^{th} level matrix of inner products[6]:

$$\det M_{(n)}(c,h) = \prod_{k=1}^{n} \psi_k(c,h)^{P(n-k)} \tag{11}$$

where

$$\psi_k(c,h) = \prod_{pq=k} (h - h_{p,q}(c)) \tag{12}$$

where p,q range over the positive integers and

(13) $$h_{p,q}(c) = \frac{[(m+1)p-mq]^2-1}{4m(m+1)} .$$

The parameter m is given by

(14) $$c = 1 - \frac{6}{m(m+1)}.$$

An overall positive constant has been suppressed in formula (11). The determinant has no zeroes in the region $1<c$, $0<h$, and the matrix of inner products for a given level is positive definite for h large enough, so there are no ghosts in the region $1\leq c$, $0\leq h$. Moreover a point is ghost free up to level n if it can be continuously connnected to $1<c$, $0<h$, without crossing a vanishing curve $h=h_{p,q}(c)$, $pq\leq n$. The region $-\infty<c<1$ corresponds to $0<m<\infty$.

The determinant is negative in some regions and positive in others. We can immediately eliminate all negative regions because they necessarily contain an odd number of ghosts. A straightforward examination of the $h_{p,q}$ curves (see the figures) shows that any point in the c,h plane for $c<1$ will have a negative determinant at some level unless that point is actually on one or more of the curves $h_{p,q}$. Then its determinant will be zero on all levels $n\geq pq$.

We already see that unitarity has limited the possible scaling dimensions for a given c (or m) to the discretely infinite set of numbers $h_{p,q}(c)$.

We mention that if the determinant vanishes at a point c,h at level n then there is a null state at that level, i.e. a state in the span of (10) orthogonal to every state including itself. Such states give rise to linear differential equations on the correlation functions[2]. The above argument tells us that correlation functions of all unitary models for $c<1$ satisfy such differential equations.

In fact the unitary theories are far more limited. We find that there are ghosts everywhere on the vanishing curves except, possibly, at a certain set of isolated points. We give a sketch of the argument here; details can be found in reference 7.

First let us examine what happens on the lowest levels. The vanishing curves for level 2 are shown in figure 2. We have already explained that region I and its border are ghost free through the second level because they are connected to $c>1$, $h>0$. The level 2 determinant is negative in region II, so there is a ghost there.

Now look at level 3 (figure 3). Region I and its border are ghost free. The level 3 determinant is negative in regions III and II_b. Region II_b was already eliminated on level 2 so the new information is that region III has a ghost. The question is whether there is a ghost on the vanishing curve $h_{2,1}$ on the border between regions II_a and III.

To settle the question we examine the pattern of subrepresentations at the intersection point A ($m=3, h=1/2$). There is a null state $|2,1\rangle$ at level 2 because $h=h_{2,1}$. There are no null states at lower levels, i.e. levels 0 and 1, so $|2,1\rangle$ is a highest weight vector, i.e. is annihilated by all the lowering operators L_n for $n>0$. It weight is given by $L_0|2,1\rangle=(h+2)|2,1\rangle$. The raising operators acting on $|2,1\rangle$ generate a subrepresentation consisting entirely of null states. In particular, $L_{-1}|2,1\rangle$ is a null state on level 3. There is also a null state $|3,1\rangle$ on level 3 because $h=h_{3,1}$ at A. The question is whether $|3,1\rangle$ is in the subrepresentation generated from $|2,1\rangle$. If it were, then Kac's determinant formula for the subrepresentation $\det M_{(3-2)}(c,h+2)=2(h+2)$ would have to vanish. Since it does not vanish, the ghost in region III is distinct from the states in the subrepresentation and eliminates all of the vanishing curve $h_{2,1}$ to the left of A. The curve $h_{3,1}$ is an example of what we will call a "cutting" curve.

Now we sketch the general argument. We consider one curve $h_{p,q}$ at a time, in order of increasing $n=pq$. We will describe how successive segments of $h_{p,q}$ are eliminated by cutting curves. The curve $h_{p,q}$ first appears as a vanishing curve at level n and remains as a vanishing curve on all levels greater than n. We define the first intersection on $h_{p,q}$ at level $n'\geqslant n$ to be the intersection closest to $c=1$ of $h_{p,q}$ with another vanishing curve. We will show that there are ghosts at all points of $h_{p,q}$ except, possibly, the first intersections. When $h_{p,q}$ appears at level n it has a first intersection (except for the case p=n, q=1). Along $h_{p,q}$ as c decreases from this first intersection the curve enters the region already known by the determinant argument to contain ghosts on previous levels, except for points where it intersects other vanishing curves from previous levels. By assumption we have already considered the points on those curves. So we need consider only the interval on $h_{p,q}$ from c=1 to the first intersection (or its whole length if q=1).

We will show that only the first intersections can be ghost free by arguing that, when a new vanishing curve $h_{p',q'}$ appears at level $n'=p'q'$ and makes a new first intersection by crossing $h_{p,q}$ between c=1 and what was the first intersection on level n'-1, the interior of the interval between the two first intersections acquires a ghost. The curve $h_{p',q'}$ thus acts as a cutting curve. Figures 4-12 show the vanishing curves on levels 4-12. Open dots mark the intervals eliminated by cutting curves.

The cutting pattern is clearest if we organize the $h_{p,q}$ curves into clusters indexed by k=p-q. An elementary argument shows that the first intersections are exactly the intersections between curves in cluster k and curves in cluster -1-k (taking the intersection closest to c=1 if there are two). These are the points

$$c = 1 - \frac{6}{m(m+1)} \qquad m=2,3,4, \cdots$$

(15)

$$h = h_{p,q}(c) = \frac{[(m+1)p-mq]^2 - 1}{4m(m+1)}$$

for $p=1,2,\cdots,m-1$ and $q=1,2,\cdots,p$.

We now generalize the argument we used for level 3. Consider the new first intersection $h=h_{p,q}=h_{p',q'}$. At level n, since $h=h_{p,q}$ and there are no vanishings at lower levels, there is a null state $|p,q\rangle$ which is a highest weight vector. It generates a highest weight representation consisting of null states. At level n' there is a null state $|p',q'\rangle$ because $h=h_{p',q'}$. It is simple to check that $\det M_{(n'-n)}(c,h+n)\neq 0$ at all the first intersections (15). It follows that

$|p',q'\rangle$ lies outside the subrepresentation generated from $|p,q\rangle$. Therefore there is a ghost on $h_{p,q}$ on one side or the other of $h_{p',q'}$. It remains only to show that the ghost is on the side away from $c=1$. We do this by following the successive first intersections the cutting curve $h_{p',q'}$ makes with curves in cluster $k=-1+q'-p'$ as it leaves the $c=1$ axis. At the first one the ghost is on the side away from $c=1$ because the other side can be connected to $c>1$. The ghost continues to exist along the cutting curve on the side farther away from $c=1$ at each successive first intersection the cutting curve makes in cluster k because there are no intervening intersections with curves not in cluster k. This serves to eliminate all the intervals on curves $h_{p,q}$ between first intersections, and leaves only the first intersections as possibly ghost free.

We have proved that all unitary representations are contained in the list (15) of first intersections, but not that all representations on the list are in fact unitary. We have verified numerically that all first intersections are ghost free through level 12. We have a heuristic argument that they remain ghost-free to all levels. Assume that the matrix of inner products can be diagonalized analytically throughout the whole region of interest ($h>0, c>0$ say), so that the norm squared of each state is a product of factors of the form $(h-h_{p,q})(h-h_{q,p})$, $p\neq q$ or $(h-h_{p,p})$. The pattern of subrepresentations[6,8] at $c=1$ implies that states whose norms vanish on the curve $h_{p,q}$ will also have zero norm on all the curves $h_{p-k,q-k}$, $k\geqslant 1$. At the first intersections m is an integer. Then the symmetry $h_{p,q}=h_{p+m,q+m+1}$ ensures that whenever a factor $(h-h_{p,q})(h-h_{q,p})$ or $(h-h_{p,p})$ is negative, another factor $(h-h_{p-k,q-k})(h-h_{q-k,p-k})$ or $(h-h_{p-k,p-k})$ is zero. Thus the absence of ghosts at the first intersections would be proved if we could verify the diagonalization assumption.

We have carried out the analytic diagonalization explicitly for levels 1–5. It might be possible to study the barrier to such a diagonalization from inconsistencies in subrepresentation patterns around polygons of vanishing curves in the c,h plane. In principle we can check for these. There are no such barriers in the region of interest for levels 1–12. Alternatively we may be able to use techniques developed by Feigin and Fuks[9], Zamolodchikov[10] and

Kadanoff and Nienhuis[11] to exhibit analytic deformations of correlation functions away from c=1.

6. CONCLUDING REMARKS

Unitarity has restricted the possible values of scaling dimensions to the simple list of rational numbers (15) when $c<1$. We display the allowed h values for m=3,4,5,6 in the right half of table I. Remember that the scaling dimension of a field is $x=h+\bar{h}$ and its spin is $h-\bar{h}$. We find[12] the Ising model is described by m=3 representation, tricritical Ising by m=4, 3-state Potts by m=5, and tricritical 3-state Potts by m=6. The known scaling indices for these models and the $h,\bar{h}$ candidates are displayed in the left half of table I. All of the known exponents are accounted for. Note that systems with continuously variable critical exponents like the Baxter, Ashkin-Teller and gaussian models live at c=1 where unitarity allows all $h\geqslant 0$.

We have used unitarity to provide strong constraints on possible reperesentations of conformal invariance, but there are additional requirements on a sensible theory: closure of the operator product expansion and crossing symmetry of correlation functions[1,2]. BPZ[2] have developed differential equation techniques which allow implementation of these conditions when the space of states is made up of representations with null states, i.e. $h=h_{p,q}(c)$. For the special values of c corresponding to m rational, $m=r/(s-r)$, $r<s$, $3r\geqslant 2s$, they have found[13] finite sets of conformal fields that must close under the operator product expansion. The scaling dimensions are given by $h=h_{p,q}(c)$, $1\leqslant p<r$, $1\leqslant q<s$. Note that the unitary representations (15) correspond to $s=r+1$. Dotsenko[14] has used the differential equation technique to find a closed operator algebra for the 3-state Potts model and to construct some of its correlation functions. Notice that in table I the 3-state Potts and tricritical 3-state Potts models do not exhaust all the possibly unitary representations. For 3-state Potts, the representations that are used make up the closed subalgebra found by Dotsenko. For the tricritical model the representations used form a closed subalgebra of the same type. It is an interesting question whether there exist larger models for m=5 and 6 using all possible

representations and containing the Potts models as sub-models.

Unitarity tells us that the only allowed representations for $c<1$ have null states, so the differential equation techniques apply. It should now be straightforward to sort out all possible unitary conformally invariant models with $c<1$. The problem is a finite one because only a finite number of representations are allowed at a given m. Such a systematic construction of scale invariant models would partially realize the bootstrap program initiated by Kadanoff[15] and Polyakov[16].

The new models with $m \geq 7$ are particularly tantalizing. Do they exist? What are they? What are their symmetries? If any of these models had a continuous internal symmetry generated by the line integral of a conserved local current then Euclidean invariance would require the two components of the current to have $h=1$, $\bar{h}=0$ and $h=0$, $\bar{h}=1$. But $h=1$ is never unitary, so there is no possibility of continuous internal symmetry.

There exists a supersymmetric extension of the Virasoro algebra called the Ramond-Neveu-Schwarz algebra[17]. Kac has written a determinant formula for this algebra as well[6]. The methods of section 5 apply; only first intersections can be ghost-free. The allowed representations, for $\frac{2}{3}c<1$, are

(16)
$$\frac{2}{3}c = 1 - \frac{8}{m(m+2)} \qquad m = 2,3,4,\cdots$$

$$h = h_{p,q}(c) = \frac{\left[\frac{1}{2}(p-q)m+p\right]^2-1}{2m(m+2)}$$

were p,q are integers, both even or both odd, $0<p<m$, $0<q\leq p$. Note that $h=1/2$ does not occur, so there are no internal supercurrents.

The representations $c=7/10$, $h=0$, $1/10$ of the super conformal algebra are composed of the representations $c=7/10$, $h=0$, $3/2$, $1/10$, $3/5$ of the Virasoro algebra. These are the representations which occur in the Z_2 invariant subalgebra of the tricritical Ising model, so it seems likely that a $c=7/10$ supersymmetric model forms a subsector of the tricritical Ising model[7]. Presumably the fermionic partners are combinations of order and disorder variables. If this identification of models is justified then physical systems described by the tricritical Ising model, such as helium adsorbed on Krypton-plated graphite[18], are supersymmetric.

Models made from representations with $m \geqslant 4$, i.e. $1 \leqslant c < 3/2$, would all be new, with the possible exception $m=4$, $c=1$. For $c=1$ the allowed representations are $h=0$, 1/16, 1/6 and 1. Models using $h=0$, 1/6 and 1 or $h=0$, 1/16 and 1 can tentatively be identified with special points in the gaussian model[7]. The $h=1$ representation would correspond to a marginal operator breaking the supersymmetry. The representations of the super algebra might also be of interest in superstring theory[19].

From a general point of view the striking feature of our result is the extreme rigidity of the conformal bootstrap. For $c<1$ it seems that unitarity is the crucial constraint. For $c \geqslant 1$ unitarity is only a mild requirement, but the possibility exists that crossing symmetry, the other element of the bootstrap, will play a crucial role in limiting the possible realizations of conformal invariance.

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TABLE I COMPARISON WITH KNOWN CRITICAL EXPONENTS[a]

model	field	x	h	$\bar{h}$	unitary highest weights $h_{p,q}(c)$[b]			
	σ	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{16}$	m=3			
Ising	ϵ	1	$\frac{1}{2}$	$\frac{1}{2}$				$\frac{1}{16}$
	ψ	$\frac{1}{2}$	$\frac{1}{2}$	0	0			$\frac{1}{2}$
	σ	$\frac{3}{40}$	$\frac{3}{80}$	$\frac{3}{80}$				
tricritical	ϵ	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{10}$	m=4			$\frac{1}{10}$
Ising		$\frac{7}{8}$	$\frac{7}{16}$	$\frac{7}{16}$		$\frac{3}{80}$		$\frac{3}{5}$
	t	$\frac{6}{5}$	$\frac{3}{5}$	$\frac{3}{5}$	0	$\frac{7}{16}$		$\frac{3}{2}$
		3	$\frac{3}{2}$	$\frac{3}{2}$				
	σ	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{1}{15}$				
	ϵ	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	m=5			$\frac{1}{8}$
3-state		$\frac{4}{3}$	$\frac{2}{3}$	$\frac{2}{3}$			$\frac{1}{15}$	$\frac{2}{3}$
Potts	CH	$\frac{9}{5}$	$\frac{7}{5}$	$\frac{2}{5}$		$\frac{1}{40}$	$\frac{21}{40}$	$\frac{13}{8}$
	ϵ'	$\frac{14}{5}$	$\frac{7}{5}$	$\frac{7}{5}$	0	$\frac{2}{5}$	$\frac{7}{5}$	3
		3	3	0				

(continued)

TABLE I (continued)

model	field	x	h	$\bar{h}$	unitary highest weights $h_{p,q}(c)$				
	σ	$\frac{2}{21}$	$\frac{1}{21}$	$\frac{1}{21}$					
	ϵ	$\frac{2}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	<u>m=6</u>				$\frac{1}{7}$
tricritical		$\frac{20}{21}$	$\frac{10}{21}$	$\frac{10}{21}$				$\frac{5}{56}$	$\frac{5}{7}$
3-state	t	$\frac{10}{7}$	$\frac{5}{7}$	$\frac{5}{7}$			$\frac{1}{21}$	$\frac{33}{56}$	$\frac{12}{7}$
Potts	CH	$\frac{17}{7}$	$\frac{12}{7}$	$\frac{5}{7}$		$\frac{1}{56}$	$\frac{10}{21}$	$\frac{85}{56}$	$\frac{22}{7}$
		$\frac{8}{3}$	$\frac{4}{3}$	$\frac{4}{3}$	0	$\frac{3}{8}$	$\frac{4}{3}$	$\frac{23}{8}$	5
		$\frac{23}{7}$	$\frac{22}{7}$	$\frac{1}{7}$					
		5	5	0					

[a] See reference 20 and references therein. For the operator of dimension 3 in the tricritical Ising model see reference 21. Note that reference 20 gives infinite sequences of possible irrelevant operators, not all of which are known to occur. We list only those which our results allow as conformal operators. The spin assignments, $h-\bar{h}$, are consistent with what is known. The CH operators can also be interpreted as derivatives of other operators.

[b] The weights $h_{p,q}$ are listed with p running horizontally from 1 to m−1 and q running vertically from 1 to p.

KEY TO THE FIGURES

The vanishing curves for level n are shown in figure n. These are the curves $h=h_{p,q}(c)$, $pq \leq n$. All of the curves in figure n are also present in figure n+1. Note that the vertical axis is $\sqrt{h}$ and that the scale changes from graph to graph.

The individual curves can be identified by their behavior for c near 1:

$$h_{p,q}(1-6\epsilon) \sim \frac{1}{4}(p-q)^2 + \frac{1}{4}(p-q)(p+q)\sqrt{\epsilon} \quad p \neq q$$

$$h_{p,p}(1-6\epsilon) \sim \frac{1}{4}(p^2-1)\epsilon.$$

The region connected to c>1, h>0 is ghost free through level n, as are the solid lines and the points marked with a circle and cross. The marked points, and also the ends of solid lines, are first intersections.

A region bounded by two solid lines and a line of open dots was ghost free on the previous level but has negative determinant and therefore a ghost on the present level.

A line of open dots represents a segment of a vanishing curve which was ghost free through the previous level but which acquires a ghost on the current level, by the cutting mechanism described in section 5.

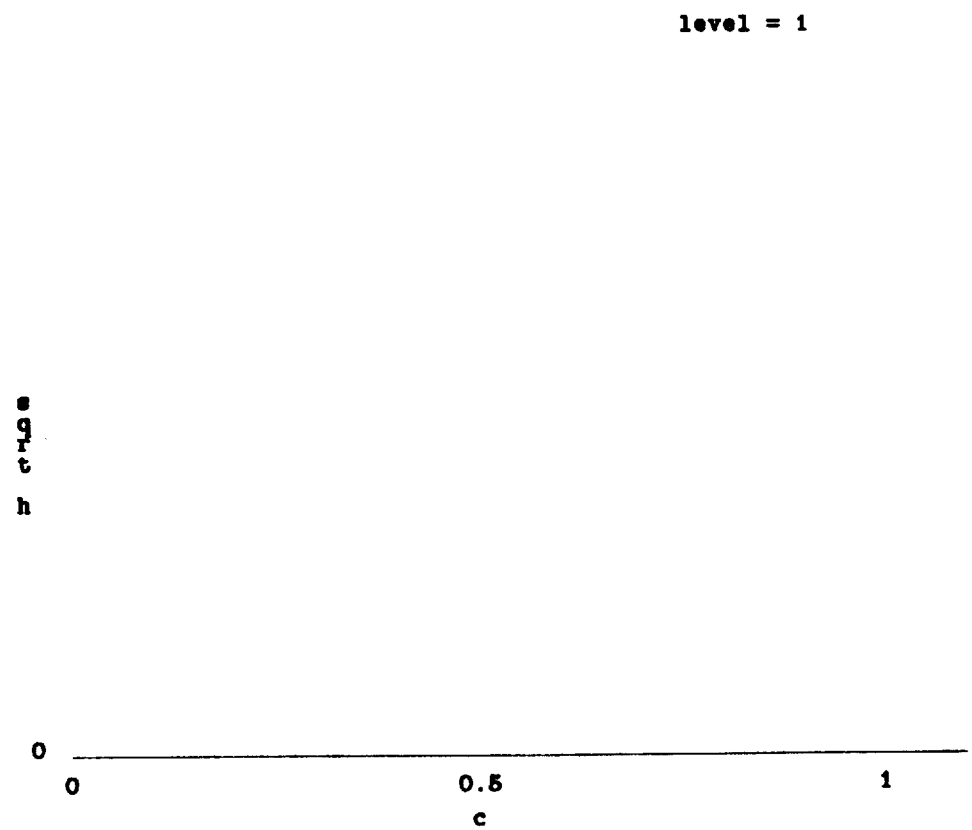

Figure 1

level = 2

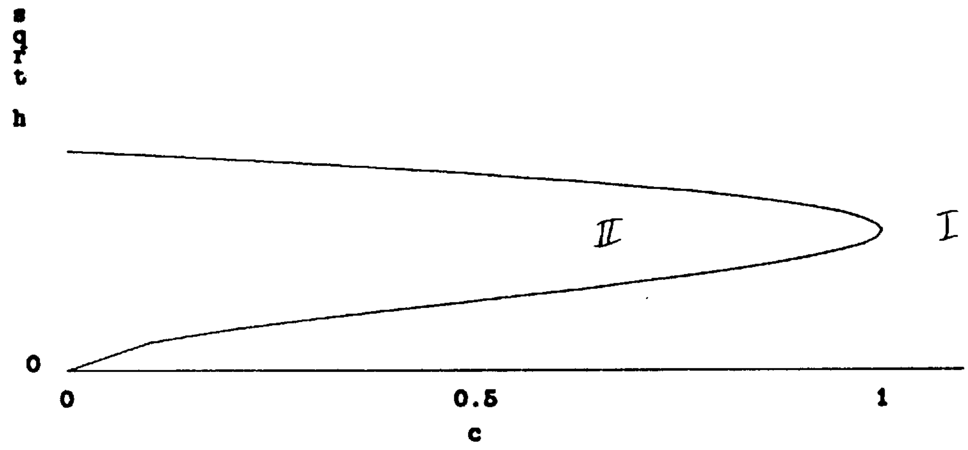

Figure 2

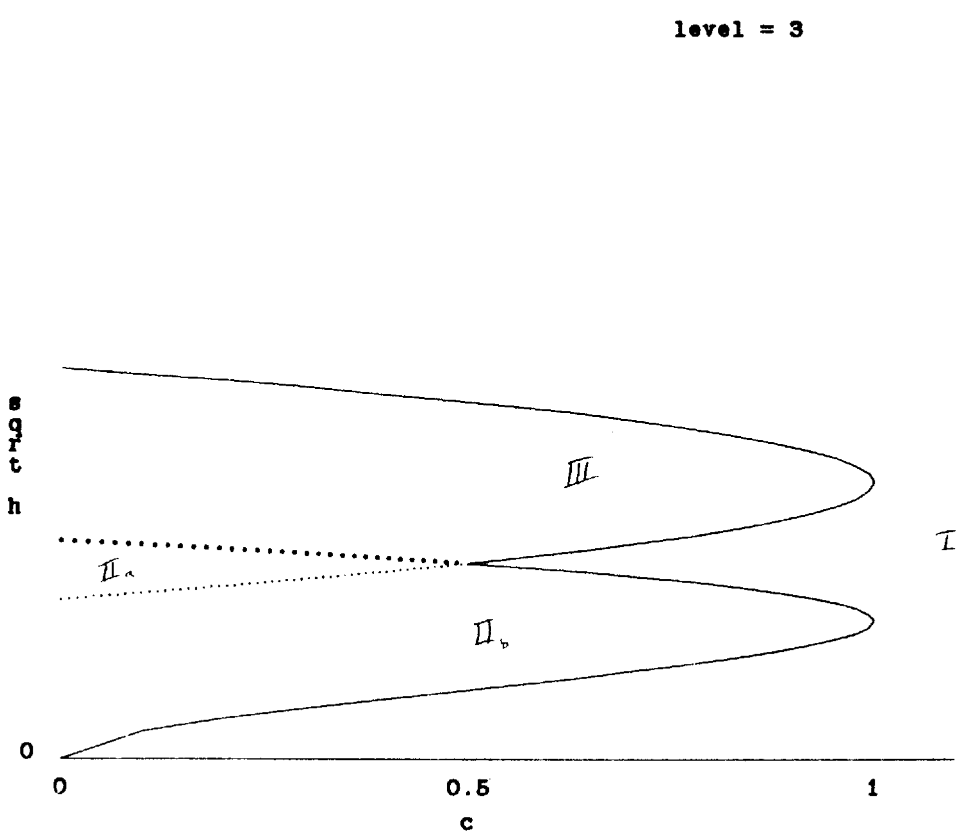

Figure 3

level = 4

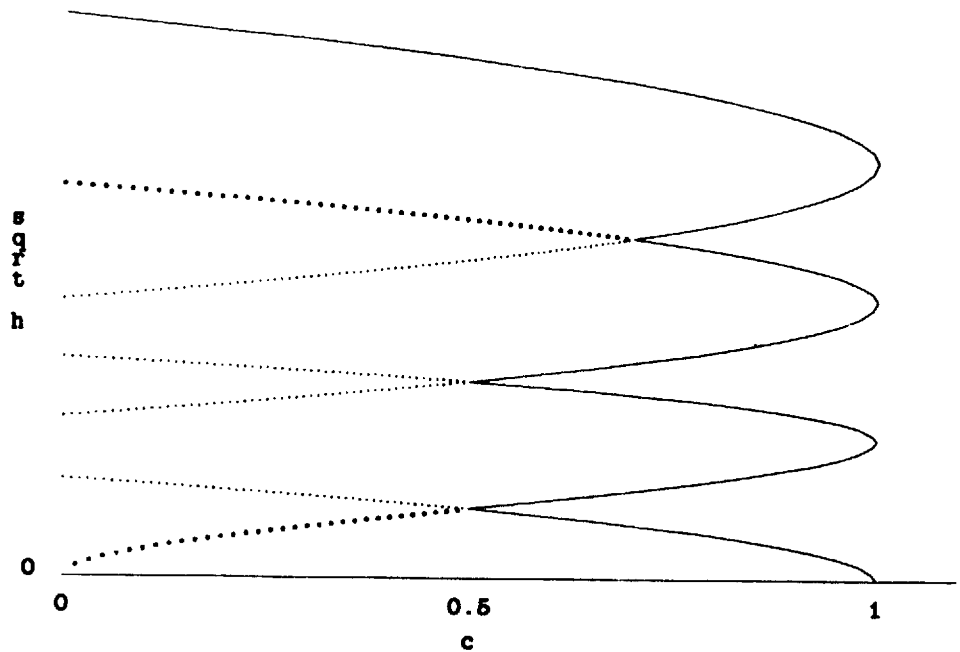

Figure 4

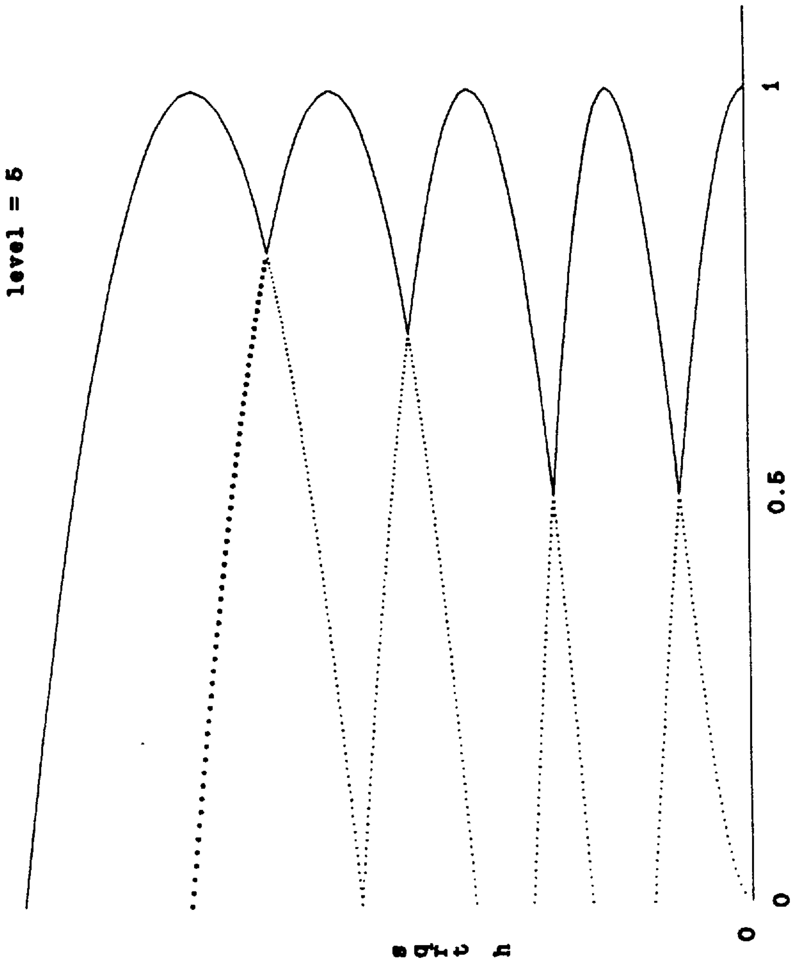

Figure 5

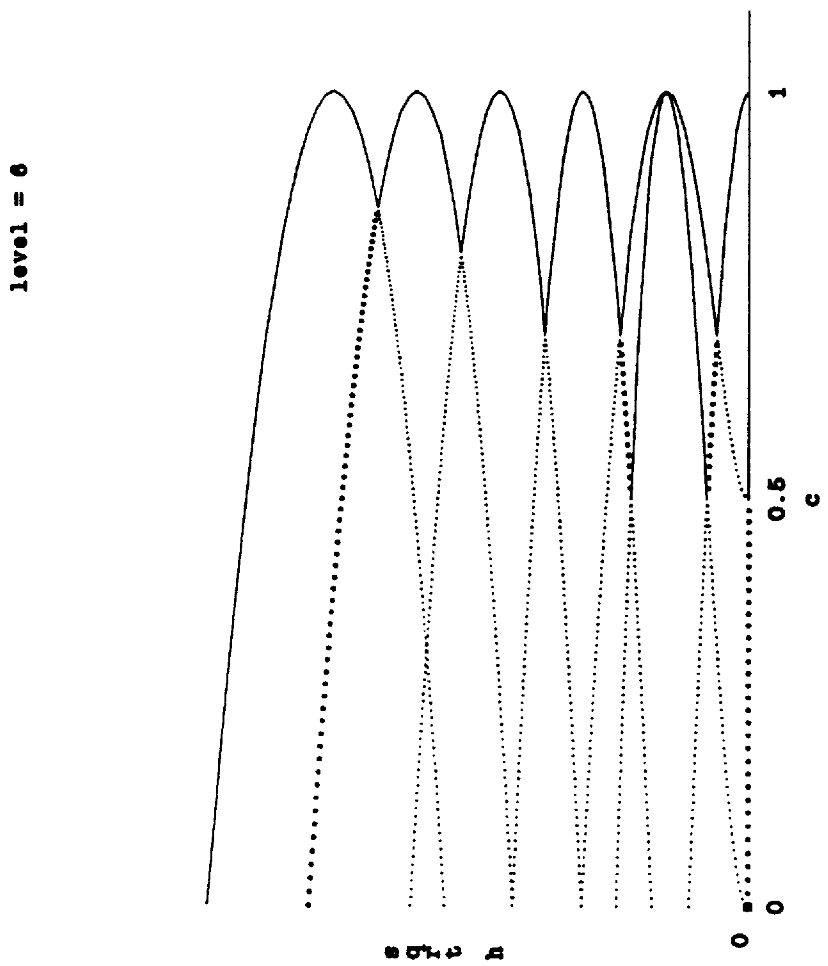

Figure 6

level = 7

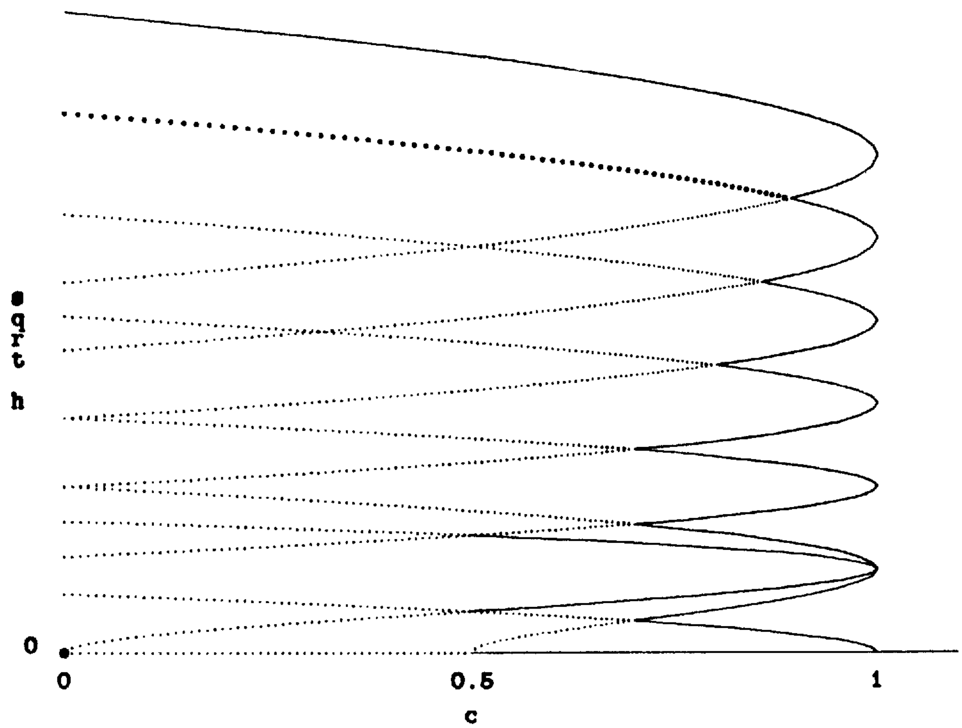

Figure 7

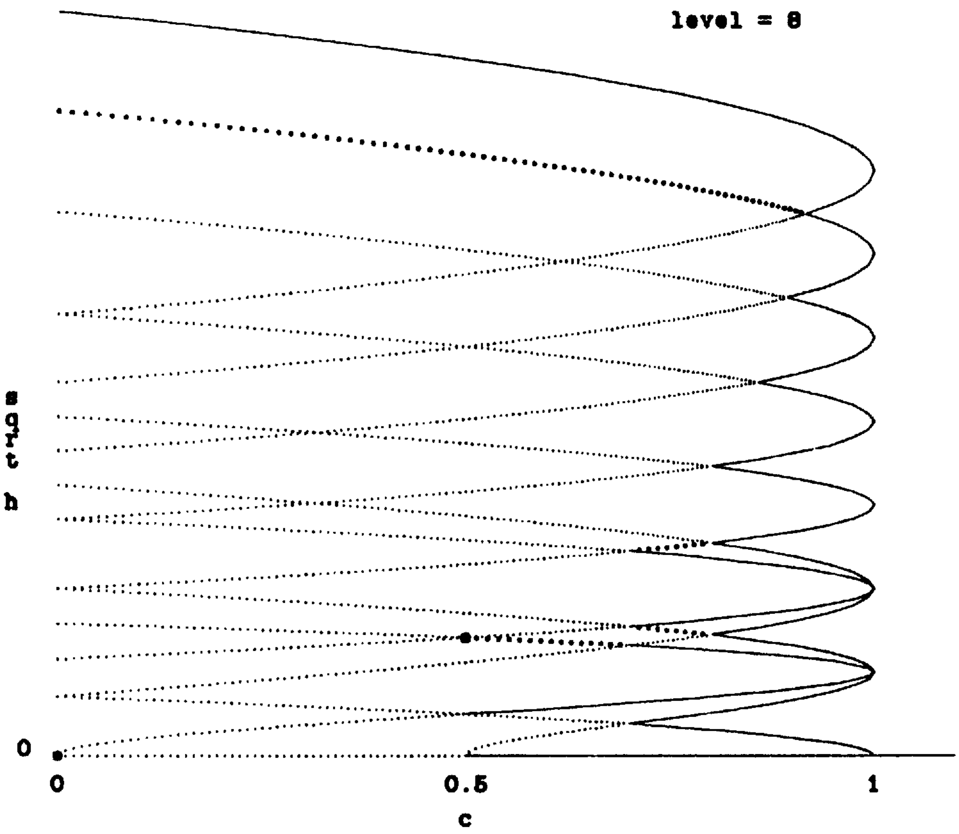

Figure 8

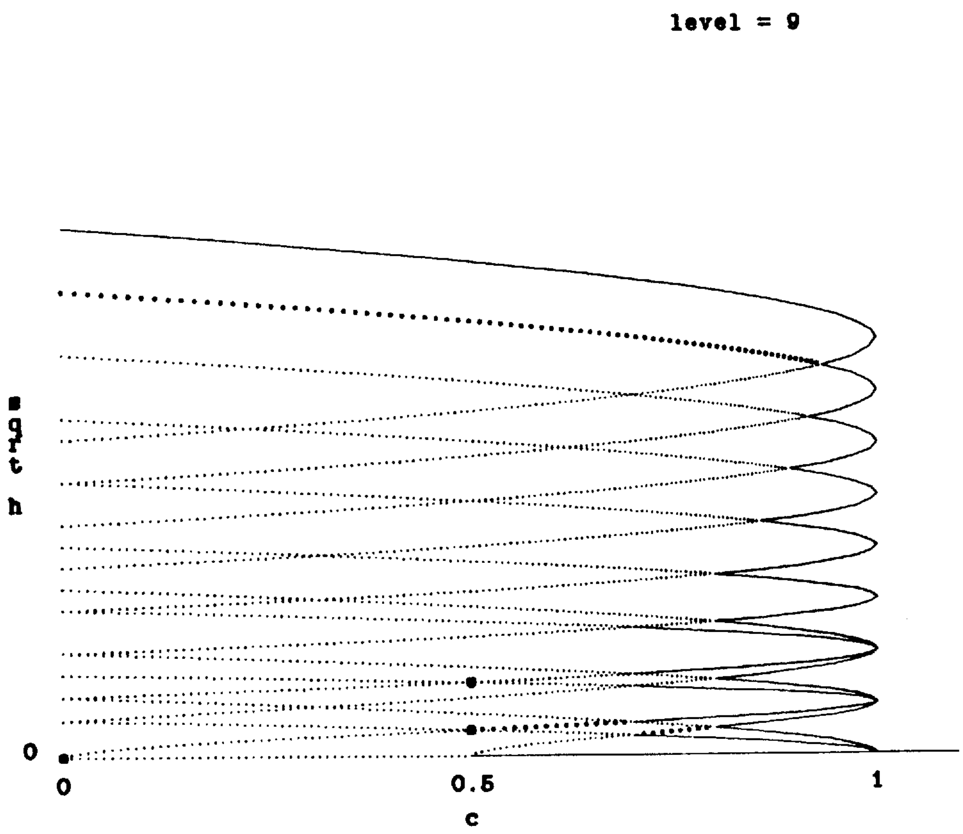

Figure 9

level = 10

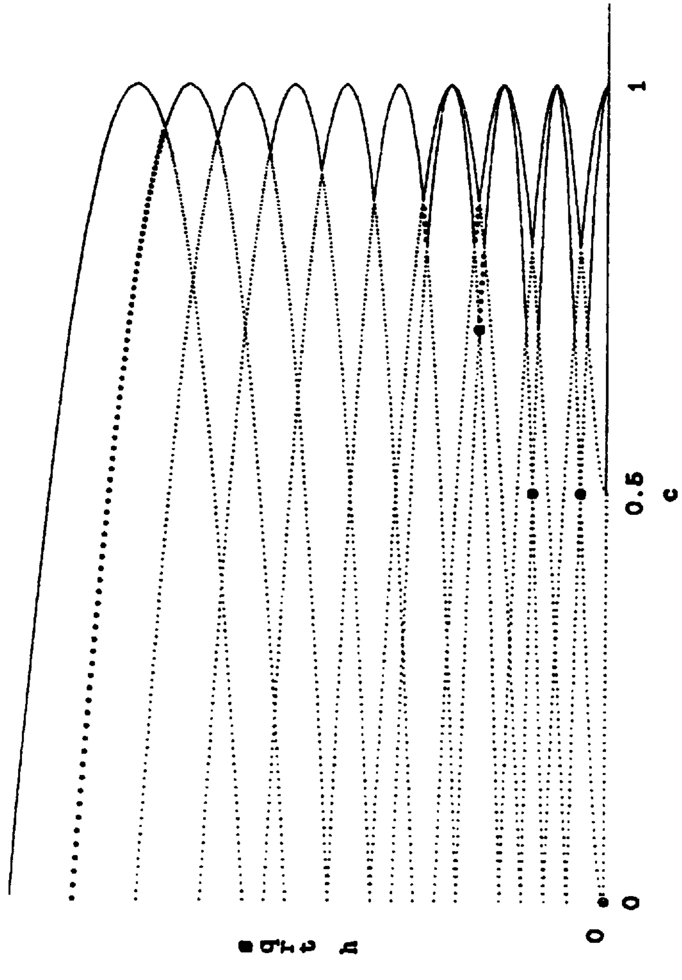

Figure 10

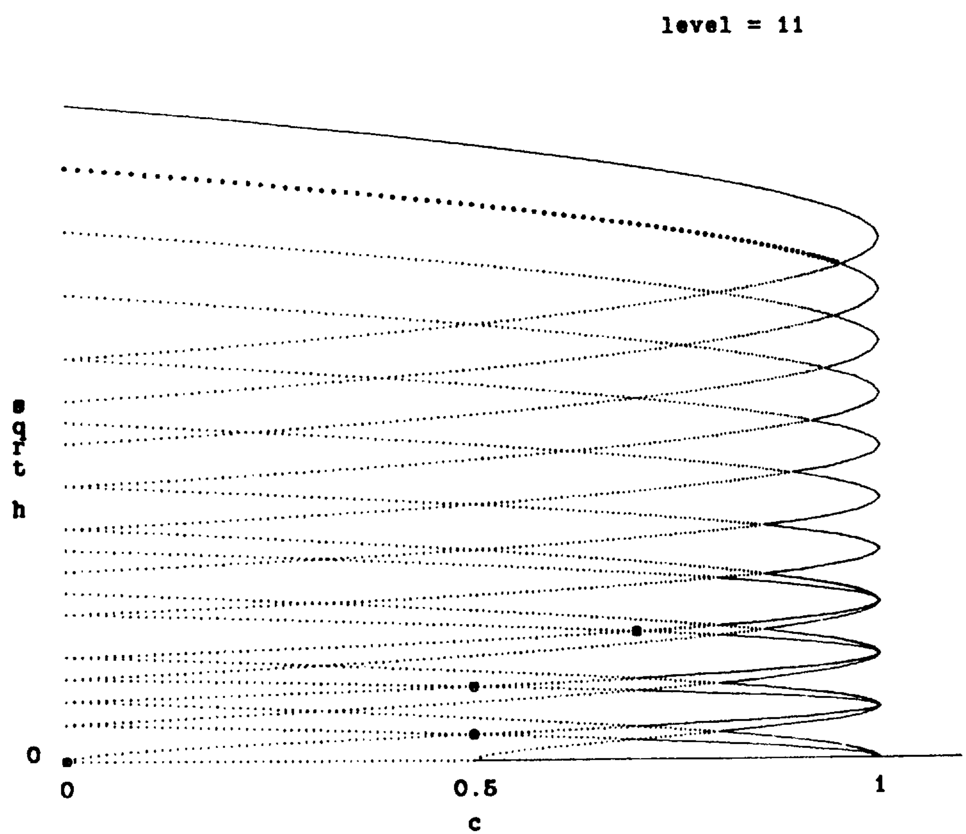

Figure 11

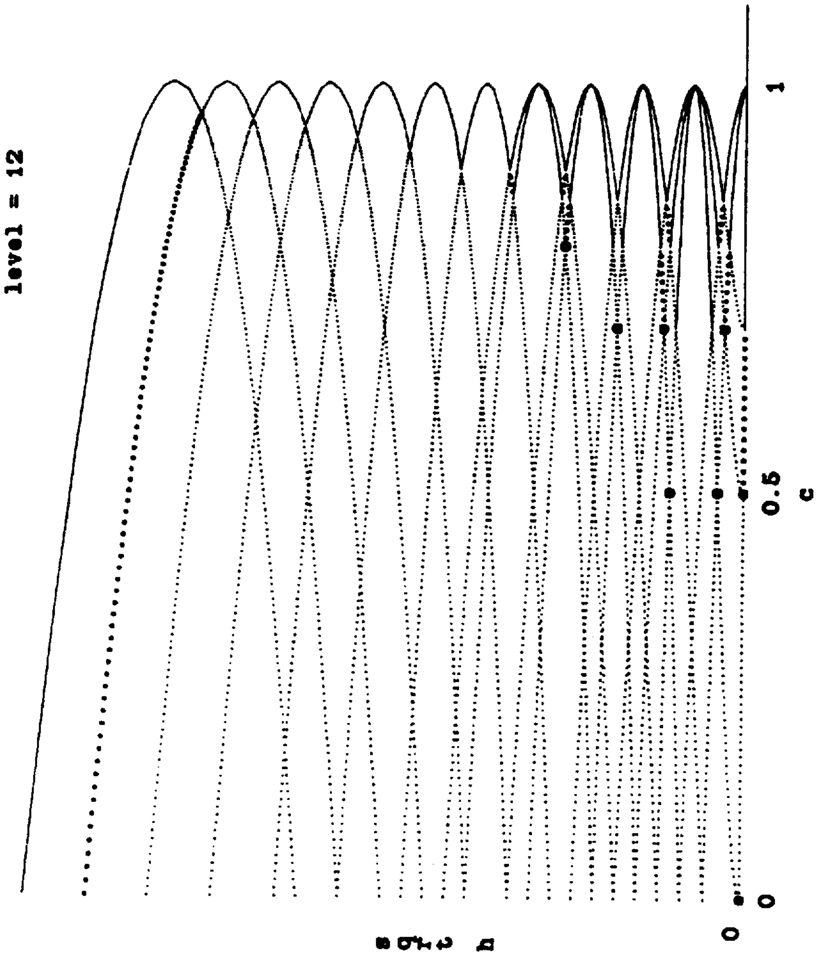

Figure 12