Quasi Riemann surfaces

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Abstract

This is a naive description of a mathematical object to be used in quantum field theory — a metric abelian group with geometric structure analogous to a Riemann surface, suitable for writing an analog of the Cauchy-Riemann equation. Examples are certain abelian groups of singular $(n-1)$-currents in a conformal $2n$-manifold. The project is to develop complex analysis on these "quasi Riemann surfaces" in analogy with ordinary Riemann surfaces, to be used to construct quantum field theories on quasi Riemann surfaces in analogy with conformal field theories on Riemann surfaces, forming a new class of constructable quantum field theories on $2n$-manifolds. The prototype is the quantum field theory of the free n-form $F(x)$ satisfying $dF = d{*}F = 0$. On the quasi Riemann surfaces it becomes the 2d conformal field theory of the free 1-form. The exposition here is unrigorous, aiming to attract interest in developing complex analysis on quasi Riemann surfaces.

Contents

1. Currents in a conformal $2n$-manifold

The goal is to formulate an analog of analysis in one complex variable on certain spaces of singular $(n{-}1)$-currents in a conformal $2n$-manifold, to be used in constructing quantum field theories on those spaces of currents in analogy with 2d conformal field theories on Riemann surfaces [1, 2].

$$\begin{aligned}
M \quad &= \text{ a smooth } 2n\text{-manifold with orientation and conformal structure}\\
&\quad\ \text{(for simplicity } n \text{ is odd, } M \text{ is compact without boundary, } \tilde{H}_{n-1}(M)=0)\\
\Omega_k^{\text{smooth}} M &= \text{ the smooth real } k\text{-forms on } M\\
\mathcal{D}_k^{\text{distr}} M &= \text{ the distributional } k\text{-currents in } M \ = \ (\Omega_k^{\text{smooth}} M)^*\\
\mathcal{D}_k^{\text{sing}} M &= \text{ the abelian group of singular } k\text{-currents in } M,\\
&\quad\ \text{generated by the } k\text{-simplices } \Delta^k \to M \text{ in } M
\end{aligned}$$

The pairing between a k-current ξ and a k-form ω is

$$\int_\xi \omega = \int_M \frac{1}{k!}\, \omega_{\mu_1\cdots\mu_k}(x)\, \xi^{\mu_1\cdots\mu_k}(x)\, d^{2n}x \tag{1.1}$$

The boundary operator on currents is dual to the exterior derivative on forms

$$\int_{\partial\xi} \omega = \int_\xi d\omega \qquad (\partial\xi)^{\mu_2\cdots\mu_k}(x) = -\partial_{\mu_1}\xi^{\mu_1\cdots\mu_k}(x) \qquad \partial^2 = 0 \tag{1.2}$$

A k-simplex σ in M is represented by a k-current $[\sigma]$

$$\sigma\colon \Delta^k \to M \qquad \int_{[\sigma]} \omega = \int_{\Delta^k} \sigma^*\omega \tag{1.3}$$

$\mathcal{D}_k^{\text{sing}} M$ is the abelian group of currents generated by the k-simplices in M, i.e., the currents representing the singular k-chains in M. The bilinear intersection form on currents

$$I_M(\xi_1,\xi_2) = \int_M \frac{1}{k_1!}\,\xi_1^{\mu_1\cdots\mu_{k_1}}\, \frac{1}{k_2!}\,\xi_2^{\nu_1\cdots\nu_{k_2}}\, \epsilon_{\mu_1\cdots\mu_{k_1}\nu_1\cdots\nu_{k_2}}\, d^{2n}x \qquad k_1+k_2 = 2n \tag{1.4}$$

is defined almost everywhere, vanishes unless $k_1+k_2=2n$, and depends only on the orientation of M (which can be written $(d^{2n}x)^{-1}\epsilon_{\mu_1\cdots\mu_{2n}}$). The intersection form satisfies

$$I_M(\xi_2,\xi_1) = (-1)^{k_1k_2} I_M(\xi_1,\xi_2) \qquad I_M(\partial\xi_1,\xi_2) = (-1)^{k_1} I_M(\xi_1,\partial\xi_2) \tag{1.5}$$

The intersection form on singular currents gives the integer intersection number (when defined). The Hodge $*$-operator on n-forms and on n-currents is conformally invariant

$$\begin{gathered}
*\omega_{\mu_1\cdots\mu_n}(x) = \frac{1}{n!}\epsilon_{\mu_1\cdots\mu_n}{}^{\nu_1\cdots\nu_n}(x)\omega_{\nu_1\cdots\nu_n}(x) \qquad \int_{*\xi}\omega = \int_\xi *\omega \qquad *^2 = (-1)^n\\
I_M(\xi,*\xi') = I_M(\xi',*\xi) \qquad I_M(\xi,*\xi) > 0 \quad \xi \neq 0 \qquad \deg(\xi) = \deg(\xi') = n
\end{gathered} \tag{1.6}$$

For n odd, $*^2=-1$ so the n-currents form a Hilbert space with complex structure $J = *$ and hermitian inner product $\langle\, \xi, \xi'\,\rangle = I_M(\xi, J\xi')$.

2. Cauchy-Riemann equation on a Riemann surface in terms of j-currents, $j = 0, 1, 2$

When $n = 1$, M is a Riemann surface. Write Σ instead of M. The augmented chain complex of singular j-currents in Σ is

$$\begin{gathered} \mathcal{D}_2^{\mathrm{sing}}\Sigma \xrightarrow{\ \partial\ } \mathcal{D}_1^{\mathrm{sing}}\Sigma \xrightarrow{\ \partial\ } \mathcal{D}_0^{\mathrm{sing}}\Sigma \xrightarrow{\ \partial\ } \mathbb{Z} \xrightarrow{\ \partial\ } 0 \\ \xi \in \mathcal{D}_0^{\mathrm{sing}}\Sigma \ \overset{\partial}{\longmapsto} \ \int_\xi 1 \end{gathered} \tag{2.1}$$

which is embedded in the augmented chain complex of distributional j-currents $\mathcal{D}_j^{\mathrm{distr}}\Sigma$.

The intersection form and the conformal Hodge $*$-operator on j-currents satisfy

$$\begin{gathered} I_\Sigma(\xi_1, \xi_2) = 0 \qquad \text{unless } j_1 + j_2 = 2 \\ I_\Sigma(\xi_2, \xi_1) = (-1)^{j_1 j_2} I_\Sigma(\xi_1, \xi_2) \qquad I_\Sigma(\partial\xi_1, \xi_2) = (-1)^{j_1} I_\Sigma(\xi_1, \partial\xi_2) \\ J = * \quad \text{on } \mathcal{D}_1^{\mathrm{distr}}\Sigma \qquad J^2 = -1 \\ \text{for } \deg(\xi) = \deg(\xi') = 1, \quad I_\Sigma(\xi, J\xi') = I_\Sigma(\xi', J\xi) \qquad I_\Sigma(\xi, J\xi) > 0 \quad \xi \neq 0 \end{gathered} \tag{2.2}$$

A fundamental solution of the Cauchy-Riemann equation

$$\begin{gathered} dz\, G(z, z')\, dz' \qquad G(z, z') = G(z', z) \\ \partial_{\bar{z}} G(z, z') = -\pi \partial_z \delta^2(z - z') \qquad G(z, z') = \frac{1}{(z - z')^2} + \text{ holomorphic} \end{gathered} \tag{2.3}$$

is determined up to products of holomorphic 1-forms. It can be regarded as a symmetric complex bilinear form on 1-currents

$$G(\xi, \xi') = \int_\xi \int_{\xi'} dz\, G(z, z')\, dz' \tag{2.4}$$

The Cauchy-Riemann equation can be expressed in terms of ∂, J, and $I_\Sigma(\xi_1, \xi_2)$

$$\begin{gathered} G(\xi, \xi') = G(\xi', \xi) \qquad G(\xi, \xi') = G(P_+\xi, P_+\xi') \qquad P_+ = \frac{1}{2}(1 + i^{-1}J) \\ G(\xi, \partial\xi_2) = -2\pi i I_\Sigma(P_+\xi, \partial\xi_2) \end{gathered} \tag{2.5}$$

3. $(n{-}1{+}j)$-currents, $j = 0, 1, 2$, in a conformal $2n$-manifold (for n odd)

For simplicity take n odd. The general case is discussed in Appendix A. For $\partial\xi_0 \in \mathcal{D}_{n-2}^{\mathrm{sing}}M$ a singular $(n{-}2)$-boundary, the abelian group

$$Q = \mathcal{D}_{n-1}^{\mathrm{sing}} M_{\mathbb{Z}\partial\xi_0} = \left\{ \xi \in \mathcal{D}_{n-1}^{\mathrm{sing}} M \colon \partial\xi \in \mathbb{Z}\partial\xi_0 \right\} \tag{3.1}$$

is to be an example of a *quasi Riemann surface*. The $\mathcal{D}_{n-1}^{\mathrm{sing}} M_{\mathbb{Z}\partial\xi_0}$ are the fibers of a bundle of quasi Riemann surfaces over $\{\mathbb{Z}\partial\xi_0\}$. There is an augmented chain complex of abelian groups Q_j analogous to (2.1)

$$\begin{gathered} Q_2 \xrightarrow{\ \partial\ } Q_1 \xrightarrow{\ \partial\ } Q \xrightarrow{\ \partial\ } \mathbb{Z}\partial\xi_0 \xrightarrow{\ \partial\ } 0 \qquad Q_0 = Q \\ Q_1 = \mathcal{D}_n^{\mathrm{sing}} M \quad Q_2 = \mathcal{D}_{n+1}^{\mathrm{sing}} M / Q^\perp \quad Q^\perp = \left\{ \xi' \in \mathcal{D}_{n+1}^{\mathrm{sing}} M \colon I_M(\xi, \xi') = 0 \ \forall \xi \in Q \right\} \end{gathered} \tag{3.2}$$

The Q_j are embedded in a chain complex of real vector spaces $Q_j^{\mathbb{R}}$ defined analogously using distributional currents in place of singular currents

$$Q_0^{\mathbb{R}} = Q^{\mathbb{R}} = \mathcal{D}_{n-1}^{\text{distr}} M_{\mathbb{R}\partial\xi_0} \qquad Q_1^{\mathbb{R}} = \mathcal{D}_n^{\text{distr}} M \qquad Q_2^{\mathbb{R}} = \mathcal{D}_{n+1}^{\text{distr}} M / Q^{\mathbb{R}\perp} \tag{3.3}$$

The intersection form $I_M(\xi_1, \xi_2)$ of M descends to a bilinear form $I_Q(\xi_1, \xi_2)$ on the Q_j with precisely the properties (2.2) suitable for writing a Cauchy-Riemann equation

$$\begin{gathered} I_Q(\xi_1, \xi_2) = 0 \qquad \text{unless } j_1 + j_2 = 2 \\ I_Q(\xi_2, \xi_1) = (-1)^{j_1 j_2} I_Q(\xi_1, \xi_2) \qquad I_Q(\partial\xi_1, \xi_2) = (-1)^{j_1} I_Q(\xi_1, \partial\xi_2) \\ J = * \quad \text{on} \quad Q_1^{\mathbb{R}} = \mathcal{D}_n^{\text{distr}} M \qquad J^2 = -1 \\ \text{for } \deg(\xi) = \deg(\xi') = 1, \quad I_Q(\xi, J\xi') = I_Q(\xi', J\xi) \qquad I_Q(\xi, J\xi) > 0 \quad \xi \neq 0 \end{gathered} \tag{3.4}$$

4. j-currents in a metric abelian group of $(n-1)$-currents

$\mathcal{D}_k^{\text{sing}} M$ is a metric space with respect to the flat metric on currents [3]. The metric completion of $\mathcal{D}_k^{\text{sing}} M$ is the abelian group of *integral* k-currents

$$\mathcal{D}_k^{\text{int}} M = (\mathcal{D}_k^{\text{sing}} M)' \qquad \mathcal{D}_k^{\text{sing}} M \subset \mathcal{D}_k^{\text{int}} M \subset \mathcal{D}_k^{\text{distr}} M \qquad \partial(\mathcal{D}_k^{\text{int}} M) \subset \mathcal{D}_{k-1}^{\text{int}} M \tag{4.1}$$

The examples $Q = \mathcal{D}_{n-1}^{\text{sing}} M_{\mathbb{Z}\partial\xi_0}$ complete to $Q' = \mathcal{D}_{n-1}^{\text{int}} M_{\mathbb{Z}\partial\xi_0}$ indexed by $\mathbb{Z}\partial\xi_0 \subset \mathcal{D}_{n-2}^{\text{int}} M$. Calculus on the metric abelian group Q' is to be based on the currents in Q' using the general construction of currents in a metric space [4]. Currents are defined as multilinear functionals of Lipschitz functions. There is no need of a smooth structure on Q'. The j-forms on Q' are to be defined as linear functions of the j-currents in Q', reversing the usual construction of currents from forms.

The equivalence $\Delta^j \times \Delta^k \simeq \Delta^{j+k}$ gives a natural map

$$\Pi_{j,k} \colon \mathcal{D}_j^{\text{sing}}(\mathcal{D}_k^{\text{int}} M) \to \mathcal{D}_{j+k}^{\text{int}} M \tag{4.2}$$

In particular,

$$\Pi_{j,n-1} \colon \mathcal{D}_j^{\text{sing}}(\mathcal{D}_{n-1}^{\text{int}} M) \to \mathcal{D}_{n-1+j}^{\text{int}} M \tag{4.3}$$

gives

$$\Pi_j \colon \mathcal{D}_j^{\text{sing}} Q' \to Q'_j \qquad j = 0, 1, 2 \tag{4.4}$$

where the Q'_j are as in (3.2). The structure (3.4) pulls back along Π_j to give a structure on the j-currents in Q' analogous to the j-currents in a Riemann surface, suitable for writing a Cauchy-Riemann equation on Q'. The project then becomes to develop complex analysis on Q' in analogy with complex analysis on Riemann surfaces.

Some of the impetus to consider this mathematical material came from comments on spaces of cycles in section 5 of [5] which referred to [6] where the maps $\Pi_{j,k}$ originate.

5. Tensor analysis on metric abelian groups

A metric abelian group is an abelian group that is complete with respect to a metric which is compatible with the group structure. Let A' be a metric abelian group, for example $A' = \mathcal{D}_k^{\text{int}} M$ or $A' = \mathcal{D}_{n-1}^{\text{int}} M_{\mathbb{Z}\partial\xi_0}$. The j-currents in A' form the vector space $\mathcal{D}_j^{\text{distr}} A'$. A j-simplex in A' is a j-current that represents a map $\sigma \colon \Delta^j \to A'$. The j-simplices generate the abelian group of singular j-currents $\mathcal{D}_j^{\text{sing}} A'$.

This section sketches a development of calculus and tensor analysis on A' based on the currents in A'. Topologies, regularity conditions, and domains of definition are left unspecified. Appendix B contains a naive remark on the need for a weak topology in addition to the metric topology on currents. Some further remarks on tensor analysis are in Appendix ??.

5.1. Tangent j-vectors as infinitesimal j-simplices

Since a j-simplex can be subdivided into arbitrarily small j-simplices, the infinitesimal j-simplices generate $\mathcal{D}_j^{\text{sing}} A'$. This is expressed by the integral representation

$$\begin{gathered} \sigma\colon \Delta^j \to A' \qquad [\sigma] = \int_{\Delta^j} \delta_{\sigma(t)} D\sigma(t)\, d^j t \\ \Delta^j = [0,1]^j \qquad \sigma_{\epsilon,t}(t') = \sigma(t+\epsilon t') \qquad \delta_{\sigma(t)} D\sigma(t) = \lim_{\epsilon \downarrow 0} \epsilon^{-j} [\sigma_{\epsilon,t}] \end{gathered} \tag{5.1}$$

$\delta_{\sigma(t)}$ is the 0-current in A' representing the point $\sigma(t)$. The limit is taken in $\mathcal{D}_j^{\text{distr}} A'$. (Appendix B comments on the limit.)

Define the space $T_j(A',\xi)$ of tangent j-vectors at $\xi \in A'$ to be the set of infinitesimal j-simplices at ξ

$$T_j(A',\xi) = \{D\sigma(0)\colon \sigma(0) = \xi\} \tag{5.2}$$

The $T_j(A',\xi)$ are the same for all ξ by translation in the abelian group A'

$$T_j(A',\xi) = T_j(A',0) \tag{5.3}$$

$T_j(A',0)$ is a vector space, not merely a cone, because A' is an abelian group and

$$D(\sigma_1 + \sigma_2) = D\sigma_1 + D\sigma_2 \qquad \text{for } \sigma_1(0) = \sigma_2(0) = 0 \tag{5.4}$$

Let $\mathcal{D}_j A'$ be the span within $\mathcal{D}_j^{\text{distr}} A'$ of the infinitesimal j-simplices

$$\mathcal{D}_j A' = \mathcal{D}_0^{\text{distr}} A' \otimes T_j(A',0) \tag{5.5}$$

Tensor analysis on A' will be based on the vector spaces $\mathcal{D}_j A'$ because the j-currents in $\mathcal{D}_j A'$ have the form of vector valued generalized functions. The integral representation (5.1) means $\mathcal{D}_j A'$ is large enough so that

$$\mathcal{D}_j^{\text{sing}} A' \subset \mathcal{D}_j A' \tag{5.6}$$

In the examples $A' = \mathcal{D}_k^{\text{int}} M$, the map $\Pi_{j,k}$ of (4.2) acts linearly on $\mathcal{D}_j A'$ via its action on the infinitesimal j-simplices. $\Pi_{j,k}$ does not act on all of $\mathcal{D}_j^{\text{distr}} A'$.

5.2. j-forms

Define the j-forms on A' to be the homomorphisms of abelian groups

$$\Omega_j A' = \operatorname{Hom}(\mathcal{D}_j^{\text{sing}} A', \mathbb{R}) \tag{5.7}$$

A homomorphism is determined by its action on the infinitesimal j-simplices so the j-forms are functions on A' with values in $T_j(A',0)^*$, i.e., sections of the j-cotangent bundle

$$\Omega_j A' = \Gamma(T_j^* A', A') \qquad T_j^* A' = A' \times T_j(A',0)^* \tag{5.8}$$

The translation invariant forms are

$$\Omega_j A'_{inv} = T_j(A',0)^* \tag{5.9}$$

6. j-currents (II)

6.1. Maps $\Pi_{j,k}\colon \mathcal{D}^{\text{sing}}_j(\mathcal{D}^{\text{int}}_k M) \to \mathcal{D}^{\text{int}}_{j+k} M$

The equivalence $\Delta^j \times \Delta^k \simeq \Delta^{j+k}$ gives

$$\Pi_{j,k}\colon \mathcal{D}^{\text{sing}}_j(\mathcal{D}^{\text{int}}_k M) \to \mathcal{D}^{\text{int}}_{j+k} M \tag{6.1}$$

$\Pi_{j,k}$ acts on infinitesimal j-simplices as a linear function

$$D\Pi_{j,k}\colon T_j(\mathcal{D}^{\text{sing}}_k M, 0) \to \mathcal{D}^{\text{distr}}_{j+k} M \tag{6.2}$$

The $\Pi_{j,k}$ are invariant under translations $T(\xi)$ in $\mathcal{D}^{\text{sing}}_k M$

$$\Pi_{0,k} T(\xi)_* = T(\xi)\Pi_{0,k} \qquad \Pi_{j,k} T(\xi)_* = \Pi_{j,k} \quad j \ge 1 \tag{6.3}$$

From

$$\partial(\Delta^j \times \Delta^k) = \partial\Delta^j \times \Delta^k + (-1)^j \Delta^j \times \partial\Delta^k \tag{6.4}$$

it follows that

$$\partial\Pi_{j,k} = \Pi_{j-1,k}\partial + (-1)^j \Pi_{j,k-1}\partial_* \tag{6.5}$$

where ∂_* is the push-forward of the boundary map $\partial\colon \mathcal{D}^{\text{int}}_k M \to \mathcal{D}^{\text{int}}_{k-1} M$

$$\partial_*\colon \mathcal{D}^{\text{sing}}_j(\mathcal{D}^{\text{int}}_k M) \to \mathcal{D}^{\text{sing}}_j(\mathcal{D}^{\text{int}}_{k-1} M) \tag{6.6}$$

6.2. Maps $\Pi_j\colon \mathcal{D}^{\text{sing}}_j Q' \to Q'_j$

Now consider $Q = \mathcal{D}^{\text{sing}}_{n-1} M_{\mathbb{Z}\partial\xi_0}$ with completion $Q' = \mathcal{D}^{\text{int}}_{n-1} M_{\mathbb{Z}\partial\xi_0}$. Define the Q'_j as in (3.2). Construct the morphism of augmented chain complexes

$$\begin{array}{ccccccccc}
\mathcal{D}^{\text{sing}}_2 Q' & \xrightarrow{\partial} & \mathcal{D}^{\text{sing}}_1 Q' & \xrightarrow{\partial} & \mathcal{D}^{\text{sing}}_0 Q' & \xrightarrow{\partial} & \mathbb{Z} & \xrightarrow{\partial} & 0 \\
\downarrow{\scriptstyle \Pi_2} & & \downarrow{\scriptstyle \Pi_1} & & \downarrow{\scriptstyle \Pi_0} & & \downarrow{\scriptstyle 1} & & \\
Q'_2 & \xrightarrow{\partial} & Q'_1 & \xrightarrow{\partial} & Q' & \xrightarrow{\partial} & \mathbb{Z} & \xrightarrow{\partial} & 0
\end{array} \tag{6.7}$$

$$\Pi_0 = \Pi_{0,n-1} \qquad \Pi_0\delta_\xi = \xi \qquad \delta_\xi \in \mathcal{D}^{\text{sing}}_0 Q' \overset{\partial}{\longmapsto} \partial\xi \in \mathbb{Z}\partial\xi_0$$

$$\Pi_1 = \Pi_{1,n-1} \qquad \Pi_2 = p_2 \circ \Pi_{2,n-1} \qquad p_2\colon \mathcal{D}^{\text{int}}_{n+1} M \to \mathcal{D}^{\text{int}}_{n+1} M / Q^\perp$$

The Π_j extend to linear maps $\Pi_j\colon \mathcal{D}_j Q' \to Q^{\mathbb{R}}_j$ forming a morphism of linear chain complexes.

The bilinear form on the Q'_j lifts to the $\mathcal{D}^{\text{sing}}_j Q'$

$$I = \Pi^* I_Q \qquad I(\xi_1, \xi_2) = I_Q(\Pi_{j_1}\xi_1, \Pi_{j_2}\xi_2) \tag{6.8}$$

The bilinear form on $\mathcal{D}^{\text{sing}}_1 Q'$ is bi-invariant under translations

$$I(T(\xi)_*\xi_1, \xi_2) = I(\xi_1, T(\xi)_*\xi_2) = I(\xi_1, \xi_2) \tag{6.9}$$

so is equivalent to a bilinear form on the tangent space $T_1(Q, 0)$.

$D\Pi_1$ identifies $T_1(Q, 0)$ with a subspace of $\mathcal{D}^{\text{distr}}_n M$

$$T_1(Q, 0) = \mathcal{V}_{1,n-1} \subset \mathcal{D}^{\text{distr}}_n M \tag{6.10}$$

$\mathcal{V}_{1,n-1}$ consists, at least roughly, of the n-currents supported on integral $(n-1)$-currents.

A crucial point is that $J = *$ should act on $T_1(Q, 0)$

$$J\mathcal{V}_{1,n-1} = \mathcal{V}_{1,n-1} \tag{6.11}$$

a germ of a proof of which is given in Appendix 1 of [2]. Given this, J lifts to

$$J\colon \mathcal{D}_1 Q' \to \mathcal{D}_1 Q' \qquad J\Pi_1 = \Pi_1 J \tag{6.12}$$

There is now structure on the currents in Q' analogous to the currents in a Riemann surface,

$$\begin{gathered} I(\xi_1,\xi_2) = 0 \qquad \text{unless } j_1+j_2 = 2 \\ I(\xi_2,\xi_1) = (-1)^{j_1 j_2} I(\xi_1,\xi_2) \qquad I(\partial\xi_1,\xi_2) = (-1)^{j_1} I(\xi_1,\partial\xi_2) \\ J\colon \mathcal{D}_1 Q' \to \mathcal{D}_1 Q' \qquad J^2 = -1 \\ I(\xi, J\xi') = I(\xi', J\xi) \qquad I(\xi,J\xi)\ge 0 \qquad \deg(\xi) = \deg(\xi') = 1 \end{gathered} \tag{6.13}$$

7. Tensor analysis (II)

Continuing discussion of the metric abelian group A, the object now is to construct a morphism of chain complexes of abelian groups

$$\begin{array}{ccccccccccccc} \cdots \xrightarrow{\partial} & \mathcal{D}^{\text{sing}}_j A' & \xrightarrow{\partial} \cdots \xrightarrow{\partial} & \mathcal{D}^{\text{sing}}_2 A' & \xrightarrow{\partial} & \mathcal{D}^{\text{sing}}_1 A' & \xrightarrow{\partial} & \mathcal{D}^{\text{sing}}_0 A' & \xrightarrow{\partial} 0 \\ & \downarrow{\scriptstyle \Pi_{(j)}} & & \downarrow{\scriptstyle \Pi_{(2)}} & & \downarrow{\scriptstyle \Pi_{(1)}} & & \downarrow{\scriptstyle \Pi_{(0)}} & \\ \cdots \xrightarrow{\partial} & A'_{(j)} & \xrightarrow{\partial} \cdots \xrightarrow{\partial} & A'_{(2)} & \xrightarrow{\partial} & A'_{(1)} & \xrightarrow{\partial} & A' & \xrightarrow{\partial} 0 \end{array} \tag{7.1}$$

embedded in a morphism of chain complexes of vector spaces

$$\begin{gathered} \begin{array}{ccccccccccccc} \cdots \xrightarrow{\partial} & \mathcal{D}_j A' & \xrightarrow{\partial} \cdots \xrightarrow{\partial} & \mathcal{D}_2 A' & \xrightarrow{\partial} & \mathcal{D}_1 A' & \xrightarrow{\partial} & \mathcal{D}_0 A' & \xrightarrow{\partial} 0 \\ & \downarrow{\scriptstyle \Pi_{(j)}} & & \downarrow{\scriptstyle \Pi_{(2)}} & & \downarrow{\scriptstyle \Pi_{(1)}} & & \downarrow{\scriptstyle \Pi_{(0)}} & \\ \cdots \xrightarrow{\partial} & A^{\mathbb{R}}_{(j)} & \xrightarrow{\partial} \cdots \xrightarrow{\partial} & A^{\mathbb{R}}_{(2)} & \xrightarrow{\partial} & A^{\mathbb{R}}_{(1)} & \xrightarrow{\partial} & A^{\mathbb{R}}_{(0)} & \xrightarrow{\partial} 0 \end{array} \\ A'_{(j)} \to A^{\mathbb{R}}_{(j)} \quad j\ge 1 \qquad A' \to A^{\mathbb{R}}_{(0)} \end{gathered} \tag{7.2}$$

The $A'_{(j)}$ and $A^{\mathbb{R}}_{(j)}$ will express the distributions $\mathcal{D}^{\text{sing}}_j A'$ and $\mathcal{D}_j A'$ modulo translations. If A were a vector space, or more generally an abelian Lie group, the boundary operators on $A'_{(j)}$ and $A^{\mathbb{R}}_{(j)}$ would be identically zero. But when A is not a vector space, the chain complexes $A'_{(j)}$ and $A^{\mathbb{R}}_{(j)}$ can express nontrivial information about A. In the examples $A = Q = \mathcal{D}^{\text{sing}}_{n-1} M_{\mathbb{Z}\partial\xi_0}$

$$Q'_{(j)} = \mathcal{D}^{\text{int}}_{j+n-1} M \qquad Q^{\mathbb{R}}_{(j)} = \mathcal{D}^{\text{distr}}_{j+n-1} M \qquad j\ge 1, \qquad Q^{\mathbb{R}}_{(0)} = \mathcal{D}^{\text{distr}}_{n-1} M_{\mathbb{R}\partial\xi_0} \tag{7.3}$$

as explained in Appendix C.

7.1. Define $A^{\mathbb{R}}_{(j)}$, $\Pi_{(j)}$, and $A'_{(j)}$ for $j\ge 1$

Using the identification $\mathcal{D}_j A' = \mathcal{D}^{\text{distr}}_0 A' \otimes T_j(A,0)$ of (5.5), define

$$A^{\mathbb{R}}_{(j)} = T_j(A,0) \qquad j\ge 1$$

$$\begin{gathered} \Pi_{(j)}\colon \mathcal{D}_j A' \to A^{\mathbb{R}}_{(j)} \qquad j\ge 1 \\ \Pi_{(j)} = \mathbf{1}^*\otimes \mathbf{1} \qquad \mathbf{1}^*\colon \mathcal{D}^{\text{distr}}_0 A' \to \mathbb{R} \qquad \mathbf{1}^*\colon \mu \mapsto \int_\mu 1 \end{gathered} \tag{7.4}$$

$$A'_{(j)} = \Pi_{(j)} \mathcal{D}^{\text{sing}}_j A' \subset A^{\mathbb{R}}_{(j)} \qquad j\ge 1$$

The translations act on $\mathcal{D}_j A'$ via $\mathcal{D}_0^{\text{distr}} A'$, so $\Pi_{(j)}$ exhibits $A^{\mathbb{R}}_{(j)}$ as $\mathcal{D}_j A'$ mod translations and $A_{(j)}$ as $\mathcal{D}_j^{\text{sing}} A'$ mod translations. So the boundary maps are well-defined. The cochain complex of invariant j-forms is dual to the chain complex $A^{\mathbb{R}}_{(j)}$

$$\Omega_j A_{inv} = (A^{\mathbb{R}}_{(j)})^* = \text{Hom}(A_{(j)}, \mathbb{R}) \qquad j \geq 1 \tag{7.5}$$

Strictly speaking, $A^{\mathbb{R}}_{(j)}$ cannot be exactly $T_j(A,0)$ but must be a completion large enough to receive the integrals $\Pi_{(j)}$ of equation (7.4).

7.2. Define $\Pi_{(0)}$ and $A^{\mathbb{R}}_{(0)}$

For $j = 0$, define

$$\Pi_{(0)} \colon \mathcal{D}_0^{\text{sing}} A' \to A' \qquad \Pi_{(0)} \colon \delta_\xi \mapsto \xi \quad \xi \in A' \tag{7.6}$$

The boundary operator $\partial \colon A_{(1)} \to A'$ is well defined because

$$\Pi_{(0)} \left[(\delta_{\xi_1} - \delta_{\xi_2}) - (\delta_{\xi+\xi_1} - \delta_{\xi+\xi_2}) \right] = 0 \tag{7.7}$$

which is to say that ∂ takes $\mathcal{D}_1^{\text{sing}} A'$ mod translations to $\mathcal{D}_0^{\text{sing}} A'$ mod $\ker \Pi_{(0)}$.

Define $A^{\mathbb{R}}_{(0)}$ by extending $\Pi_{(0)}$ from $\mathcal{D}_0^{\text{sing}} A'$ to $\mathcal{D}_0 A'$

$$A^{\mathbb{R}}_{(0)} = \mathcal{D}_0 A' / \text{span} \left\{ \delta_{\xi_1+\xi_2} - \delta_{\xi_1} - \delta_{\xi_2},\, \xi_1, \xi_2 \in A' \right\} \tag{7.8}$$

so that

$$(A^{\mathbb{R}}_{(0)})^* = \text{Hom}(A, \mathbb{R}) \tag{7.9}$$

The cochain complex of invariant forms on A is modified in degree 0

$$\cdots \xrightarrow{d} \Omega_2 A_{inv} \xrightarrow{d} \Omega_1 A_{inv} \xrightarrow{d} (A^{\mathbb{R}}_{(0)})^* \xrightarrow{d} 0 \tag{7.10}$$

7.3. A necessary condition on A

The abelian groups $A'_{(j)}$ are embedded by design in the vector spaces $A^{\mathbb{R}}_{(j)}$ for $j \geq 1$. But A' is not necessarily embedded in $A^{\mathbb{R}}_{(0)}$. There is a morphism $A' \to A^{\mathbb{R}}_{(0)}$ but there is no guarantee that the morphism is injective. For example, if $A = U(1)$ then $A^{\mathbb{R}}_{(0)} = 0$. On the other hand, in the examples $A = Q = \mathcal{D}_{n-1}^{\text{sing}} M_{\mathbb{Z}\partial\xi_0}$ the morphism $A' \to A^{\mathbb{R}}_{(0)}$ is the embedding $\mathcal{D}_{n-1}^{\text{int}} M_{\mathbb{Z}\partial\xi_0} \to \mathcal{D}_{n-1}^{\text{distr}} M_{\mathbb{R}\partial\xi_0}$. The condition that $A' \to A^{\mathbb{R}}_{(0)}$ be injective must be imposed by hand on the metric abelian group A.

7.4. Augmentation

When there is an augmentation $A \xrightarrow{\partial} \mathbb{Z}$, the augmentation can be regarded as a function on A' that can be integrated over a 0-current, giving augmentations of the chain complexes in (7.1) and (7.2).

$$\mathcal{D}_0 A' \xrightarrow{\partial} \mathbb{R} \qquad \mathcal{D}_0^{\text{sing}} A' \xrightarrow{\partial} \mathbb{Z} \qquad \delta_\xi \xmapsto{\partial} \partial\xi \quad \xi \in A' \tag{7.11}$$

8. Definition of quasi Riemann surface

QRS-1 *Let Q be a metric abelian group with a morphism $Q \xrightarrow{\partial} \mathbb{Z}$, the augmentation.*

The currents in Q are to have structure equivalent to (6.13) which is sufficient for a Cauchy-Riemann equation. The examples are $Q = \mathcal{D}^{\mathrm{sing}}_{n-1} M_{\mathbb{Z}\partial\xi_0}$ for M a conformal $2n$-manifold. When n=1 this is $Q = \mathcal{D}^{\mathrm{sing}}_0 \Sigma$ for $M = \Sigma$ a Riemann surface.

The structure is formulated on the chain complex of vector spaces $Q^{\mathbb{R}}_{(j)}$ from (7.2) with $A = Q$ in analogy with the chain complex $\mathcal{D}^{\mathrm{distr}}_j \Sigma$ of currents in a Riemann surface and the chain complex (3.2) of $(j+n-1)$-currents in a $2n$-manifold M. Then the structure is lifted to the currents $\mathcal{D}_j Q$ in Q via the morphisms $\Pi_{(j)}$.

8.1. Structure on the tangent space $Q^{\mathbb{R}}_{(1)} = T_1(Q, 0)$

QRS-2 *There should be a complex structure J and an antisymmetric bilinear form $I_Q(\xi_1, \xi_2)$ on $T_1(Q, 0)$.*

(1) *$I_Q(\xi_1, J\xi_2)$ should be symmetric, hermitian, and positive definite, making $T_1(Q, 0)$ a Hilbert space (which should be infinite-dimensional and separable).*

(2) *$T_1(Q, 0)$ should decompose into three orthogonal subspaces*

$$\begin{aligned} T_1(Q,0) &= \operatorname{Im}\partial \oplus T_1(Q,0)_H \oplus J \operatorname{Im}\partial \\ T_1(Q,0)_H &= \operatorname{Ker}\partial \cap J \operatorname{Ker}\partial \end{aligned} \tag{8.1}$$

(3) *The harmonic subspace $T_1(Q, 0)_H$ should be finite dimensional, or at least $\operatorname{Im}\partial$ should be infinite dimensional. In the examples, $T_1(Q, 0) = \mathcal{D}^{\mathrm{distr}}_n M$ and $T_1(Q, 0)_H$ is the finite dimensional vector space of harmonic n-currents.*

8.2. Assume connectedness and nondegeneracy

For simplicity make two assumptions on Q.

(1) connectedness: The reduced homology $\tilde{H}_0(Q) = 0$, i.e., $\partial Q_{(1)} = \operatorname{Ker}\partial$.

In the examples $Q = \mathcal{D}^{\mathrm{sing}}_{n-1} M_{\mathbb{Z}\partial\xi_0}$ this is $\tilde{H}_{n-1}(M) = 0$. For $n = 1$, the Riemann surface $M = \Sigma$ is connected. For $n > 1$, the $2n$-manifold M has $H_{n-1}(M) = 0$.

(2) nondegeneracy: $\partial Q = \mathbb{Z}$

In the examples, this is $\partial\xi_0 \neq 0$.

8.3. Extend $I_Q(\xi_1, \xi_2)$ to $Q^{\mathbb{R}}_{(0)} \times Q^{\mathbb{R}}_{(2)}$

$I_Q(\xi_1, \xi_2)$ extends to $\partial Q^{\mathbb{R}}_{(1)} \times Q^{\mathbb{R}}_{(2)}$ and $Q^{\mathbb{R}}_{(2)} \times \partial Q^{\mathbb{R}}_{(1)}$ by the conditions

$$I_Q(\partial\xi_1, \xi_2) = (-1)^{j_1} I_Q(\xi_1, \partial\xi_2) \qquad I_Q(\xi_2, \xi_1) = (-1)^{j_1 j_2} I_Q(\xi_1, \xi_2) \tag{8.2}$$

but additional structure is needed to specify the extension to all of $Q^{\mathbb{R}}_{(0)} \times Q^{\mathbb{R}}_{(2)}$. By the two assumptions above, there exists $\xi_0 \in Q$ with $\partial\xi_0 = 1$ and $Q^{\mathbb{R}}_{(0)} = \mathbb{R}\xi_0 \oplus \operatorname{Im}\partial$. So the extension is completely specified once $\xi_2 \mapsto I_Q(\xi_0, \xi_2)$ is given.

QRS-3 *There should be* $\omega(\xi_0) \in T_2(Q,0)^* = \Omega_2 Q_{inv}$ *for some* $\xi_0 \in Q$ *with* $\partial\xi_0 = 1$, *depending on the choice of* ξ_0 *by*

$$\begin{aligned} &\omega(\xi_0 + \partial\xi_1) = \omega(\xi_0') + d\eta(\xi_1) \qquad \xi_1 \in Q^{\mathbb{R}}_{(1)} \\ &\eta(\xi_1) \in T_1(Q,0)^* \text{ is given by } \int_{\xi_1'} \eta(\xi_1) = I_Q(\xi_1', \xi_1) \quad \xi_1' \in Q^{\mathbb{R}}_1 \end{aligned} \tag{8.3}$$

$\omega(\xi_0)$ *should not vanish on* $\operatorname{Ker}\partial \subset Q^{\mathbb{R}}_{(2)}$.

The extension of $I_Q(\xi_1,\xi_2)$ to $Q^{\mathbb{R}}_{(0)} \times Q^{\mathbb{R}}_{(2)}$ is now completely specified by

$$I_Q(\xi_0,\xi_2) = \int_{\xi_2} \omega(\xi_0) \qquad \xi_2 \in Q^{\mathbb{R}}_{(2)} \tag{8.4}$$

8.4. $Q^{\mathbb{R}}_j$ and Q'_j

The vector spaces $Q^{\mathbb{R}}_j$ as in (3.2) are

$$\begin{aligned} Q^{\mathbb{R}} = Q^{\mathbb{R}}_0 = Q^{\mathbb{R}}_{(0)} \qquad Q^{\mathbb{R}}_1 = Q^{\mathbb{R}}_{(1)} \\ Q^{\mathbb{R}}_2 = Q^{\mathbb{R}}_{(2)}/(Q^{\mathbb{R}}_{(0)})^{\perp} \qquad (Q^{\mathbb{R}}_{(0)})^{\perp} = \operatorname{Ker}\partial \cap \operatorname{Ker}\omega(\xi_0) \end{aligned} \tag{8.5}$$

The abelian groups $Q_j \subset Q^{\mathbb{R}}_j$ are the images of the $\mathcal{D}^{\text{sing}}_j Q'$.

8.5. Integrality condition

QRS-4 $I_Q(\xi_1,\xi_2)$ *should take integer values on* $Q'_{(1)}$ *and* $\omega(\xi_0')$ *should take integer values on* $Q'_{(2)}$, *both wherever defined (which should be almost everywhere).*

$$Q'_{(j)} = \Pi_{(j)}\mathcal{D}^{\text{sing}}_j Q' \subset Q^{\mathbb{R}}_{(j)} \tag{8.6}$$

8.6. Lift to the currents in Q

Define

$$\begin{aligned} \mathcal{D}^{\mathbb{R}}_0 &= \mathcal{D}^{\text{dist}}_0 Q' &&= \mathcal{D}_0 Q' \qquad \mathcal{D}_0 = \mathcal{D}^{\text{sing}}_0 Q' = \mathcal{D}_0 Q' \\ \mathcal{D}^{\mathbb{R}}_1 &= \mathcal{D}^{\text{dist}}_0 Q' \otimes T_1(Q,0) &&= \mathcal{D}_1 Q' \\ \mathcal{D}^{\mathbb{R}}_2 &= \mathcal{D}^{\text{dist}}_0 Q' \otimes Q^{\mathbb{R}}_2 \end{aligned}$$

so $\mathcal{D}^{\mathbb{R}}_2$ is a quotient of $\mathcal{D}_2 Q' = \mathcal{D}^{\text{dist}}_0 Q' \otimes Q^{\mathbb{R}}_{(2)}$.

Lift $I_Q(\xi_1,\xi_2)$ to a bilinear form $I(\xi_1,\xi_2)$ on the currents in Q by

$$I(\xi_1,\xi_2) = I_Q(\Pi_{(j_1)}\xi_1, \Pi_{(j_2)}\xi_2) \qquad \xi_1 \in \mathcal{D}_{j_1}Q', \quad \xi_2 \in \mathcal{D}_{j_2}Q', \tag{8.7}$$

Lift J to act on $\mathcal{D}_1 Q'$ via the identification $\mathcal{D}_1 A' = \mathcal{D}^{\text{distr}}_0 A' \otimes T_1(A,0)$ of (5.5).

The integrality condition implies that $I(\xi_1,\xi_2)$ will take integer values on $\mathcal{D}^{\text{sing}}_j Q'$

8.7. Some basic consequences

$\omega(\xi_0)$ restricted to $\operatorname{Ker}\partial \subset Q^{\mathbb{R}}_{(0)}$ does not depend on the choice of ξ_0. It descends to a nonzero form on the one-dimensional $\operatorname{Ker}\partial \subset Q^{\mathbb{R}}_2$ so $[Q] \in Q_2$ is uniquely defined by

$$[Q] \in Q_2 \qquad \partial[Q] = 0 \qquad \int_{[Q]} \omega(\xi_0) = 1 \tag{8.8}$$

$[Q]$ has the property

$$I_Q(\xi, [Q]) = \partial\xi \qquad \xi \in Q \tag{8.9}$$

There are decompositions

$$\begin{aligned} &Q^{\mathbb{R}}_0 = \mathbb{R}\xi_0 \oplus [Q]^{\perp} && \operatorname{Im}\partial = [Q]^{\perp} \qquad [Q]^{\perp} \cong J\partial Q^{\mathbb{R}}_2 \\ &Q^{\mathbb{R}}_1 = \partial Q^{\mathbb{R}}_2 \,\oplus\, T_1(Q,0)_H \,\oplus\, J\partial Q^{\mathbb{R}}_2 && \\ &Q^{\mathbb{R}}_2 = \mathbb{R}[Q] \oplus \xi_0^{\perp} && \operatorname{Ker}\partial = \mathbb{R}[Q] \qquad \xi_0^{\perp} \cong \partial Q^{\mathbb{R}}_2 \end{aligned} \tag{8.10}$$

In the examples $Q = \mathcal{D}^{\text{sing}}_0\Sigma$, $[Q]$ is the 2-current representing the Riemann surface Σ. For $Q = \mathcal{D}^{\text{sing}}_{n-1}M_{\mathbb{Z}\partial\xi_0}$ with $n > 1$, $[Q]$ is a certain equivalence class of $(n{+}1)$-boundaries in M

$$[Q] = \left\{\partial\xi \in \mathcal{D}^{\text{sing}}_{n+1}M \colon I_M(\xi_0, \partial\xi) = 1\right\} \tag{8.11}$$

The set of such $\partial\xi$ should be non-empty which is an additional condition on $\partial\xi_0$, an irreducibility condition. To accommodate all $\partial\xi_0 \in \mathcal{D}^{\text{sing}}_{n-1}M$ the nondegeneracy assumption $\partial Q = \mathbb{Z}$ should be dropped. Then $\partial Q = N\mathbb{Z}$ for some integer $N \geq 0$ and $\omega(\xi_0)$ depends on a choice of $\xi_0 \in Q$ with $\partial\xi_0 = N$. The integrality condition becomes $\omega(\xi_0)(Q'_{(2)}) = N\mathbb{Z}$.

8.8. Cauchy-Riemann equation

All the properties of (6.13) are now satisfied. This structure is sufficient to write a Cauchy-Riemann equation on the quasi Riemann surface Q analogous to the Cauchy-Riemann equation (2.5) on a Riemann surface

$$\begin{gathered} G\colon \mathcal{D}_1Q \otimes \mathcal{D}_1Q \to \mathbb{C} \qquad G(\xi,\xi') = G(\xi',\xi) \\ G(\xi,\xi') = G(P_+\xi, P_+\xi') \qquad P_+ = \frac{1}{2}(1 + i^{-1}J) \\ G(\xi, \partial\xi_2) = -2\pi i I(P_+\xi, \partial\xi_2) \end{gathered} \tag{8.12}$$

In addition, the translation invariance of $I(\xi_1,\xi_2)$ and J allows translation invariance to be imposed on the solution

$$G(T(\xi'')_*\xi,\, \xi') = G(\xi,\, T(\xi'')_*\xi') = G(\xi,\xi') \tag{8.13}$$

NOTE: that the equation is really solved on the $Q^{\mathbb{R}}_j$.
using the decomposition
introduce period lattice
For use in the linear morphism section.

9. Morphisms and quasi holomorphic curves

A linear morphism $F\colon Q_A \to Q_B$ of quasi Riemann surfaces consists of linear operators $F_j\colon Q^{\mathbb{R}}_{Aj} \to Q^{\mathbb{R}}_{Bj}$ which preserve ∂, J, and $I_Q(\xi_1,\xi_2)$, including

$$F_{-1}\colon \mathbb{R} \to \mathbb{R} \qquad F_{-1}\partial = \partial F_0 \tag{9.1}$$

A morphism will be a linear morphism satisfying some integrality and regularity conditions. For simplicity assume that Q_A and Q_B are nondegenerate and connected as in section 8.2.

9.1. Linear morphisms, isomorphisms, automorphisms

By the decomposition (8.10) F_1 consists of two partial unitary operators

$$\begin{aligned} F_{1,\mathrm{exact}}\colon \partial Q^{\mathbb{R}}_{A2} &\to \partial Q^{\mathbb{R}}_{B2} & F^{\dagger}_{1,\mathrm{exact}} F_{1,\mathrm{exact}} &= 1 \\ F_{1,\mathrm{harmonic}}\colon T_1(Q_A,0)_H &\to T_1(Q_B,0)_H & F^{\dagger}_{1,\mathrm{harmonic}} F_{1,\mathrm{harmonic}} &= 1 \end{aligned} \tag{9.2}$$

F_0 on $\operatorname{Im}\partial \subset Q^{\mathbb{R}}_{A0}$ is determined by $F_{1,\mathrm{exact}}$. Then F_0 is completely determined by

$$\xi'_{B0} = F_0 \xi_{A0} \in Q^{\mathbb{R}}_B \tag{9.3}$$

ξ'_{B0} also determines F_{-1} by

$$F_{-1} 1 = \partial \xi'_{B0} \in \mathbb{R} \tag{9.4}$$

F_2 must satisfy

$$\begin{aligned} \partial F_2[Q_A] &= F_2 \partial [Q_A] = 0 \\ I_{Q_B}(F_0\xi_{A0}, F_2[Q_A]) &= I_{Q_A}(\xi_{A0}, [Q_A]) = 1 \end{aligned} \tag{9.5}$$

so

$$F_{-1}1 = \partial \xi'_{B0} \neq 0 \qquad F_2[Q_A] = (F_{-1}1)^{-1}[Q_B] \tag{9.6}$$

Finally,

$$F_2(\xi^{\perp}_{A0}) \subset (\xi'_{B0})^{\perp} \tag{9.7}$$

the last of which is equivalent to

$$\partial(F_2\xi_2) - F_1(\partial\xi_2) \in (F_1 Q_{A1})^{\perp} \tag{9.8}$$

so F_2 is determined up to an arbitrary linear operator

$$F_{2,0}\colon \xi^{\perp}_{A0} \to (F_{1,\mathrm{exact}} \partial Q^{\mathbb{R}}_{A2})^{\perp} \cap (\partial Q^{\mathbb{R}}_{B2}) \tag{9.9}$$

Thus a linear morphism is given by

(1) the partial unitary operators $F_{1,\mathrm{exact}}$ and $F_{1,\mathrm{harmonic}}$
(2) $\xi'_{B0} \in Q^{\mathbb{R}}_B$, $\partial\xi'_{B0} = 1$
(3) the linear operator $F_{2,0}$

Strictly, $\partial\xi'_{B0}$ can be any nonzero real number, but sending $\xi'_{B0} \to a\xi'_{B0}$, $a \in \mathbb{R}$, $a \neq 0$ produces an essentially equivalent morphism. So $\partial\xi'_{B0} = 1$ which is $F_2[Q_A] = [Q_B]$ might as well be assumed.

A linear isomorphism has $F_{1,\mathrm{exact}}$ and $F_{1,\mathrm{harmonic}}$ both unitary, implying $F_{2,0} = 0$. The linear automorphism group is thus the semi-direct product

$$\operatorname{Aut}(Q) = Q^{\mathbb{R}}_{0,0} \rtimes (UQ^{\mathbb{R}}_{1,\mathrm{harmonic}} \times UQ^{\mathbb{R}}_{1,\mathrm{exact}}) \qquad Q_{0,0} = \operatorname{Ker}\partial = \operatorname{Im}\partial \subset Q \tag{9.10}$$

where $Q_{0,0}$ acts on Q by translation, i.e., $\xi_B = \xi_A + \partial\xi'_A$, $\partial\xi'_A \in Q_{0,0}$.

9.2. Integrality and regularity conditions

A morphism might be defined as a linear morphism that preserves the integral structure $Q_j \subset Q^{\mathbb{R}}_j$ and also the metric structure on Q. However, the Cauchy Riemann equation sees only ∂, J, and $I(\xi_1,\xi_2)$. These are preserved by every linear morphism. So it might be reasonable to impose only one additional condition: that the linear morphism preserve the period lattice $L \subset Q^{\mathbb{R}}_{1,\mathrm{harmonic}}$, i.e., $F_{1,\mathrm{harmonic}}(L_A) \subset L_B$. With this definition of morphism, a quasi Riemann surface is classified up to isomorphism by the genus $g = \dim_{\mathbb{C}}(Q^{\mathbb{R}}_{1,\mathrm{harmonic}})$ and the rank $2g$ lattice L in the complex Hilbert space $Q^{\mathbb{R}}_{1,\mathrm{harmonic}} \cong \mathbb{C}^g$.

9.3. Quasi holomorphic curves

A quasi holomorphic curve is a morphism $C\colon Q(\Sigma) \to Q$ where Σ is a Riemann surface and $Q(\Sigma) = \mathcal{D}_0^{\mathrm{sing}}\Sigma$ is the quasi Riemann surface associated to Σ. A solution of the Cauchy Riemann equation on Q pulls back along C to an ordinary solution on the Riemann surface Σ. A local quasi holomorphic curve is one where Σ is the open complex disk.

Complex analysis on Q should be equivalent to ordinary analysis in one complex variable on each of a suitable collection of (local) quasi holomorphic curves, subject to compatibility conditions when quasi holomorphic curves overlap. The physics application would be to construct a quantum field theory on Q as an ordinary two-dimensional conformal field theory on each of those quasi holomorphic curves.

10. *

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Appendix A. General case (n even or odd)

Now consider the general case of M a conformal $2m$-manifold with n even or odd. For n even, the conformal Hodge $*$-operator on n-forms satisfies $*^2 = (-1)^n = 1$, and the intersection form $I_M(\xi,\xi')$ on n-currents is symmetric. So the construction of a quasi Riemann surface for general n requires the complex currents

$$\mathcal{D}^{\mathrm{distr}}_{k,\mathbb{C}}M = (\mathcal{D}^{\mathrm{distr}}_{k}M)\otimes\mathbb{C} \tag{A.1}$$

Write

$$\deg(\xi) = n-1+\deg'(\xi) \qquad k = n-1+j \qquad \xi\in\mathcal{D}^{\mathrm{distr}}_{k,\mathbb{C}}M \tag{A.2}$$

Choose a root ϵ_n of the equation

$$\epsilon_n^2 = (-1)^{n-1} \tag{A.3}$$

then define

$$J = \epsilon_n * \quad \text{on } \mathcal{D}^{\mathrm{distr}}_{n,\mathbb{C}}M \tag{A.4}$$

$$I_M\langle\,\bar\xi_1,\xi_2\,\rangle = \epsilon_n^{-1}\,(-1)^{(n-1)k_2}\,I_M(\bar\xi_1,\xi_2) \quad \text{on } \overline{\mathcal{D}^{\mathrm{distr}}_{k_1,\mathbb{C}}M}\otimes\mathcal{D}^{\mathrm{distr}}_{k_2,\mathbb{C}}M \tag{A.5}$$

satisfying

$$I_M\langle\,\bar\xi_1,\xi_2\,\rangle = 0 \ \text{ unless } j_1+j_2 = 2 \tag{A.6}$$

$$I_M\langle\,\bar\xi_1,\xi_2\,\rangle = (-1)^{j_1j_2}\overline{I_M\langle\,\bar\xi_2,\xi_1\,\rangle} \qquad I(\partial\bar\xi_1,\xi_2) = (-1)^{j_1}I(\bar\xi_1,\partial\xi_2) \tag{A.7}$$

$$J\in\mathrm{End}(\mathcal{D}^{\mathrm{distr}}_{n,\mathbb{C}}M) \qquad J^2 = -1 \tag{A.8}$$

$$I_M\langle\,\bar\xi,J\xi'\,\rangle = \overline{I_M\langle\,\bar\xi',J\xi\,\rangle} \qquad I_M\langle\,\bar\xi,J\xi\,\rangle > 0 \quad \xi\neq 0 \qquad \deg'(\xi)=\deg'(\xi')=1 \tag{A.9}$$

For each $\partial\xi_0 = \partial\mathcal{D}^{\mathrm{sing}}_{n-1}M$ there is the augmented chain complex of abelian groups/complex vector spaces

$$\begin{array}{ccccccccccc} 0 & \xrightarrow{\partial} & Q_2 & \xrightarrow{\partial} & Q_1 & \xrightarrow{\partial} & Q & \xrightarrow{\partial} & \mathbb{Z} & \xrightarrow{\partial} & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \xrightarrow{\partial} & Q_2^{\mathbb{C}} & \xrightarrow{\partial} & Q_1^{\mathbb{C}} & \xrightarrow{\partial} & Q^{\mathbb{C}} & \xrightarrow{\partial} & \mathbb{C} & \xrightarrow{\partial} & 0 \end{array} \tag{A.10}$$

$$\begin{aligned} Q_{-1} &= \mathbb{Z}\partial\xi_0 & Q^{\mathbb{C}}_{-1} &= \mathbb{C} \\ Q = Q_0 &= (\mathcal{D}^{\text{sing}}_{n-1} M_{\mathbb{Z}\partial\xi_0}) \oplus i(\partial\mathcal{D}^{\text{sing}}_n M) & Q^{\mathbb{C}}_0 &= \mathcal{D}^{\text{distr}}_{n-1,\mathbb{C}} M_{\mathbb{C}\partial\xi_0} \\ Q_1 &= (\mathcal{D}^{\text{sing}}_n M) \oplus i(\mathcal{D}^{\text{sing}}_n M) & Q^{\mathbb{C}}_1 &= \mathcal{D}^{\text{distr}}_{n,\mathbb{C}} M \\ Q_2 &= [(\mathcal{D}^{\text{sing}}_{n+1} M) \oplus i(\mathcal{D}^{\text{sing}}_{n+1} M)]/Q_0^{\perp} & Q^{\mathbb{C}}_2 &= (\mathcal{D}^{\text{distr}}_{n+1,\mathbb{C}} M)/(Q^{\mathbb{C}}_0)^{\perp} \end{aligned} \tag{A.11}$$

Q is defined so that $\partial(Q) \subset \mathbb{Z}$, so that $\tilde{H}_0(Q) = 0$ in the connected case, and so that $T_0(Q, 0)$ is the complex vector space $\mathcal{V}_{1,n-1} \otimes \mathbb{C}$ on which $J = \epsilon_n *$ acts.

A *complex* quasi Riemann surface is a abelian group Q with augmentation and metric completion and also with an involution $\xi \mapsto \bar{\xi}$, $\partial\bar{\xi} = \partial\xi$, called complex conjugation. The definition is as before, but with $\mathbb{C}$ in place of $\mathbb{R}$ and with a sesquilinear form $I_Q\langle\, \bar{\xi}_1, \xi_2 \,\rangle$ in place of the bilinear form $I_Q(\xi_1, \xi_2)$.

Appendix B. Topologies for infinitesimal j-simplices

The construction of an infinitesimal j-simplex as a derivative in (5.1) requires taking a limit of j-currents $\lim_{\epsilon\downarrow 0} \epsilon^{-j}[\sigma_{\epsilon,t}]$. Suppose $A = \mathbb{R}^d$ as a simple example. In the ordinary sense of derivative, $D\sigma(t)$ is a j-vector in $\mathbb{R}^d$ at $\sigma(t)$. The j-currents $\epsilon^{-j}[\sigma_{\epsilon,t}]$ converge weakly to $\delta_{\sigma(t)} D\sigma(t)$ but they do not converge in the metric topology. The limit in the metric topology is a 0-current $|v|\delta_{\hat{v}}$ in the unit sphere of j-vectors at $\sigma(t)$, where $v = D\sigma(t)$, $\hat{v} = v/|v|$. The linear map $a\delta_{\hat{v}} \mapsto a\hat{v}$ projects the metric j-tangent space down to the weak j-tangent space. This example suggests that the infinitesimal j-simplices in a general abelian A must be constructed as weak limits of currents. The same weak topology would be used to define j-forms as homomorphisms from $\mathcal{D}^{\text{sing}}_j A$ to $\mathbb{R}$ as in (5.7). In the same vein, a weak topology is needed for the vector space of infinitesimal elements of A so that in the examples $A = \mathcal{D}^{\text{sing}}_k M$ the result will be a subspace of $\mathcal{D}^{\text{distr}}_k M$. The metric topology gives much too large a space of infinitesimals.

B.1. Derivatives $D\Pi_{j,k} \colon T_j(\mathcal{D}^{\text{sing}}_k M, 0) \to \mathcal{D}^{\text{distr}}_{j+k} M$

$\Pi_{j,k}$ acts on infinitesimal j-simplices as a linear function

$$D\Pi_{j,k} \colon T_j(\mathcal{D}^{\text{sing}}_k M, 0) \to \mathcal{D}^{\text{distr}}_{j+k} M \tag{B.1}$$

The image is

$$\mathcal{V}_{j,k} = D\Pi_{j,k}(T_j(\mathcal{D}^{\text{sing}}_k M, 0)) \qquad \mathcal{V}_{j,k} \subset \mathcal{D}^{\text{distr}}_{j+k} M \tag{B.2}$$

$\mathcal{V}_{j,k}$ is, roughly, the subspace of $(j+k)$-currents supported on integral k-currents. The boundary operator ∂ is injective on $\mathcal{V}_{j,k}$

$$(\operatorname{Ker} \partial) \cap \mathcal{V}_{j,k} = \{0\} \tag{B.3}$$

There is another linear function

$$T_j(\mathcal{D}^{\text{sing}}_k M, 0) \xrightarrow{D\partial} T_j(\mathcal{D}^{\text{sing}}_{k-1} M, 0) \xrightarrow{D\Pi_{j,k-1}} \mathcal{D}^{\text{distr}}_{j+k-1} M \tag{B.4}$$

Together, the two linear functions $D\Pi_{j,k}$ and $D\Pi_{j,k-1} \circ D\partial$ identify

$$T_j(\mathcal{D}^{\text{sing}}_k M, 0) = \partial(\mathcal{V}_{j,k}) \oplus \mathcal{V}_{j,k-1} \tag{B.5}$$

Appendix C. Tangent spaces

In particular,

$$\Pi_{0,k}\delta_\xi = \xi \tag{C.1}$$

Appendix D. *

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